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A METHOD OF DETERMINING THE EFFECT OF AIRPLANE STABILITY ON THE GUST LOAD FACTOR

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SUMMARY

The equations of motion of a conventional airplane penetrating a gust are given in detail for determining the effects due to stability on the gust load factor. A convenient numerical method is derived in matrix notation which affords a systematic procedure of solving the equations for a unit jump or arbitrary forcing function. The solution for the airplane motions for the stick-free condition is modified by neglecting the elevator terms so that the effects under similar stability conditions may be calculated with the stick in a fixed position. If the complete response of the airplane is known, wing and tail loads may be computed fairly conveniently. Sample calculations were performed to illustrate the application of the equations.

INTRODUCTION

In the past the attempts to study the most effective stability parameters in a gust have been hampered by the complexity of the equations and the time consumed in their solution. In numerous analyses, the pitch effects are either considered empirically or neglected entirely. A method of calculation that would be feasible, would give reasonably accurate results, and would require only a moderate expenditure of time is, therefore, needed for analytical studies of airplane motions in gusts.

Analytical solutions by means of operators have been made for calculating the motions of a wing penetrating a sharp-edge gust (see reference 1). An extension of this method for solving the more complicated equations would be impractical. In reference 2, calculations were made by an iterative method for obtaining the solutions of the equations of motion in two degrees of freedom of a canard airplane. For this analysis the downwash effects could be neglected and the equations thereby considerably simplified.

The present analysis gives an application of the calculus of finite differences for solving the equations of motion of airplanes in gusty air. The numerical solutions have been derived in matrix notation from the equations of motion in three degrees of freedom - vertical, pitch, and elevator displacement - forward-speed variations being neglected, and sample calculations are made. For a complete understanding of the method, a working knowledge of matrix operations is required.

SYMBOLS

L	lift, pounds
C_L	lift coefficient
α	angle of attack, radians
$\frac{dC_L}{d\alpha}$	slope of lift curve, per radian
A	aspect ratio
ρ	mass density of air, slugs per cubic foot
U	gust velocity, feet per second
V	forward velocity, feet per second
q	dynamic pressure $\left(\frac{1}{2}\rho V^2\right)$
g	acceleration due to gravity, feet per second per second
S	area of surface, square feet
c	reference chord, feet
W	airplane weight, pounds
μ_g	mass parameter $\left(\frac{2W}{g \frac{dC_L}{d\alpha} S\rho c}\right)$
s_1	nondimensional distance penetrated into gust, chords

s	variable of integration
Δs	increment of s used in matrix solution
m	variable in recurrence formula $\left(\frac{s}{\Delta s}\right)$
τ	nondimensional discontinuity taken to nearest integer due to lag of tail penetration $\left(\frac{l_{wt} + c}{c\ \Delta s}\right)$
C_{L_g}	normalized unsteady-lift function due to penetration of a sharp-edge gust
Φ	normalized unsteady-lift function due to penetration of an arbitrary-shape gust
$C_{L\epsilon_g}$	equivalent normalized unsteady-lift function on tail due to downwash caused by wing penetration into a sharp-edge gust
Φ_ϵ	equivalent normalized unsteady-lift function on tail due to downwash caused by wing penetration of an arbitrary-shape gust
C_{L_α}	normalized unsteady-lift function for a unit jump of angle of attack
$C_{L\epsilon_\alpha}$	equivalent normalized unsteady-lift function on tail due to downwash caused by a unit jump of angle of attack on wing
ϵ_g	normalized unsteady-downwash angle at horizontal tail due to penetration of wing into a sharp-edge gust
ξ	normalized unsteady-downwash angle at horizontal tail due to penetration of wing into an arbitrary-shape gust
ϵ_α	normalized unsteady-downwash angle at horizontal tail due to unit jump of angle of attack on wing
B	transformed unsteady-lift function C_{L_α} for wing $\left(1 - C_{L_{\alpha_W}}\right)$
C	transformed unsteady-lift function C_{L_α} for tail $\left(1 - C_{L_{\alpha_t}}\right)$
Δn	load-factor increment encountered by airplane, multiples of acceleration due to gravity
θ	pitch angle, radians

δ elevator-deflection angle, radians

l_w horizontal distance from center of gravity of airplane to aerodynamic center of wing, positive when aerodynamic center is ahead of center of gravity, feet

l_t horizontal distance from center of gravity of airplane to aerodynamic center of tail, feet

l_{wt} distance between trailing edge of wing mean aerodynamic chord to leading edge of tail mean aerodynamic chord, feet

l_ϵ lag for downwash angle to be effective on tail when wing undergoes a unit jump of angle of attack $\left(\frac{\frac{l_{wt}}{c} + 0.23}{\Delta s}\right)$

l_{C_L} lag for unsteady-lift function due to downwash to be effective on tail when wing undergoes a unit jump of angle of attack $\left(\frac{\frac{l_{wt}}{c} + 0.80}{\Delta s}\right)$

l_h distance between airplane center of gravity and elevator hinge, feet

η tail-efficiency factor $\left(\frac{q_t}{q}\right)$

$\frac{d\epsilon}{d\alpha}$ steady-downwash angle per unit jump of angle of attack on wing

$\frac{d\epsilon_1}{d\alpha}$ asymptotic value of downwash angle per unit jump of angle of attack on wing including effects due to unsteady lift on tail

I pitching moment of inertia of airplane about center of gravity, slug-feet^2

I_e moment of inertia of elevator about its hinge, slug-feet^2

I_s moment of inertia of control stick about its pivot, slug-feet^2

H	total hinge moment, positive when moment tends to depress trailing edge
H_e	gravitational moment of elevator about its hinge, positive when moment tends to depress trailing edge, slug-feet
H_s	gravitational moment of control stick about its pivot, positive when moment tends to depress trailing edge, slug-feet
r	control gearing ratio, angular stick deflection divided by angular elevator deflection
K	viscous damping constant, in control system, pounds per foot per second
Q	ratio of damping velocity to elevator velocity, feet per second per radian per second
C_{h_α}	elevator hinge-moment coefficient due to angle-of-attack change on tail; floating tendency is positive when surface floats against relative wind
C_{h_δ}	elevator hinge-moment coefficient due to elevator deflection; restoring tendency is positive when surface is over-balanced
$\frac{d\alpha_e}{d\delta}$	elevator-effectiveness factor
$\left(\frac{\partial C_L}{\partial D\delta}\right)_A$	part of additional lift due to angular velocity of elevator caused by acceleration of potential flow
$\left(\frac{\partial C_L}{\partial D\delta}\right)_B$	part of additional lift due to angular velocity of elevator caused by effective increase in camber
$\left(\frac{\partial C_h}{\partial D\delta}\right)_A$	part of hinge moment due to angular velocity of elevator caused by acceleration of potential flow
$\left(\frac{\partial C_h}{\partial D\delta}\right)_B$	part of hinge moment due to angular velocity of elevator caused by effective increase in camber

Operational symbols:

D, D^2	differential operators $\left(\frac{d}{ds}, \frac{d^2}{ds^2}\right)$

I^y single integration of arbitrary function $y(s)$

II^y double integration of arbitrary function $y(s)$

Matrix symbols:

E, F, G constant elements in matrix solution

$\left\{ \quad \right\}$ column matrix representing variables in recurrence formula

$\left\{ \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix} \right\}$ column matrix representing the forcing function due to gust in recurrence formula

$[M]$ square matrix representing inverse of the matrix coefficients of Δn, $D^2\theta$, and $D^2\delta$ at point m

$[S_1]$, . . . $[S_6]$ rectangular matrices representing constant elements in recurrence formula

$\lfloor \quad \rfloor$ row matrix

$[\quad]_0$ matrices representing constant elements in recurrence formula for stick-fixed condition

Subscripts:

e elevator

max maximum

w wing

t tail

DERIVATION OF METHOD

Differential Equations of Motion

The evaluation of the effects of stability on the gust load factor may be determined from an analysis of the motions of the airplane under various stability conditions. In setting up the differential equations, the following assumptions are made:

(a) The airplane is free to move vertically and to pitch about its lateral axis and has an elevator restrained only by the force produced by the viscous friction in the control system.

(b) The airplane is in steady level flight before entering the gust and has initially no displacements, velocities, and accelerations with regard to its degrees of freedom.

(c) The forward speed is constant.

(d) The forces producing the hinge moment on the elevator are steady-state forces.

(e) The over-all aerodynamic force on the whole tail is a transient force.

An equation of equilibrium exists between inertia and aerodynamic forces and moments for each degree of freedom. In figure 1, the forces and moments acting on a conventional airplane are shown. All distances, forces, and moments are shown in a positive direction. The three equations representing the incremental forces and moments become (see fig. 1)

$$W\ \Delta n = W \frac{d^2 Z}{dt^2}\frac{1}{g} = L_W + L_t = W\ \Delta n_W + W\ \Delta n_t \tag{1}$$

$$I \frac{V^2}{c^2} D^2\theta = W l_W\ \Delta n_W - W l_t\ \Delta n_t \tag{2}$$

$$I_e \frac{V^2}{c^2}\left(D^2\theta + D^2\delta\right) - H_e\left(g\ \Delta n - l_h \frac{V^2}{c^2} D^2\theta\right) + r\left[I_s \frac{V^2}{c^2}\left(r\ D^2\delta - D^2\theta\right) - H_s g\ \Delta n\right] = H \tag{3}$$

The transient aerodynamic forces produced by the airplane penetrating the gust and the forces due to its motion relative to an equilibrium position may be appropriately described by a set of integral equations. That part of the force due to the penetration of the gust is independent of the motions and is therefore analogous to the forcing function acting on any dynamic system. The forces produced for each degree of freedom must be accounted for separately. These individual forces are described by a number of Duhamel's integrals. If superposition is valid, the individual forces may be summed to give the total force acting on the body due to its motion. These aerodynamic forces may be converted to their corresponding load-factor increments. Accordingly, for the motion of the wing, the load-factor increment Δn_w is described by the following integral equation:

$$\Delta n_w(s_1) = K_1 \int_0^{s_1} C_{L_{g_w}}(s_1 - s) D \frac{U}{U_{max}} ds - K_4 \int_0^{s_1} C_{L_{\alpha_w}}(s_1 - s) \Delta n(s) ds + K_7 \int_0^{s_1} C_{L_{\alpha_w}}(s_1 - s) D\theta ds - K_{10} \int_0^{s_1} C_{L_{\alpha_w}}(s_1 - s) D^2\theta ds \tag{4}$$

where the coefficients of the integrals are

$$K_1 = \left(\frac{dC_L}{d\alpha}\right)_w \frac{\rho}{2} VS \frac{U_{max}}{W} \tag{4a}$$

$$K_4 = \left(\frac{dC_L}{d\alpha}\right)_w \frac{\rho}{2} \frac{S}{W} cg \tag{4b}$$

$$K_7 = \left(\frac{dC_L}{d\alpha}\right)_w \frac{S}{W} \frac{\rho}{2} V^2 \tag{4c}$$

$$K_{10} = \left(\frac{dC_L}{d\alpha}\right)_w \frac{S}{W} \frac{\rho}{2} V^2 \frac{l_w}{c} \tag{4d}$$

The expression for the forces acting on the tail correspond to the terms describing the wing together with the forces caused by the interaction of the wing and the tail. This last effect, called downwash, introduces additional Duhamel's integrals which are functions of the motions of the wing. The summations of the integrals represent the total aerodynamic forces on the tail and may be converted to load-factor increments as follows:

$$\Delta n_t(s_1) = K_2 \int_0^{s_1} C_{L_{g_t}}(s_1 - s) D \frac{U}{U_{max}} ds + K_3 \int_0^{s_1} C_{L_{\epsilon_g}}(s_1 - s) D \frac{U}{U_{max}} ds -$$

$$K_5 \int_0^{s_1} C_{L_{\alpha_t}}(s_1 - s)\Delta n(s)\ ds - K_6 \int_0^{s_1} C_{L_{\epsilon_\alpha}}(s_1 - s)\Delta n(s)\ ds +$$

$$K_8 \int_0^{s_1} C_{L_{\alpha t}}(s_1 - s)D\theta\ ds + K_9 \int_0^{s_1} C_{L_{\epsilon_\alpha}}(s_1 - s)D\theta\ ds +$$

$$K_{11} \int_0^{s_1} C_{L_{\alpha t}}(s_1 - s)D^2\theta\ ds - K_{12} \int_0^{s_1} C_{L_{\epsilon_\alpha}}(s_1 - s)D^2\theta\ ds +$$

$$K_{13} \int_0^{s_1} C_{L_{\alpha t}}(s_1 - s)D\delta\ ds + K_{14} \int_0^{s_1} C_{L_{\alpha t}}(s_1 - s)D^2\delta\ ds \qquad (5)$$

where the coefficients of the integrals are

$$K_2 = K_1 \frac{\left(\frac{dC_L}{d\alpha}\right)_t}{\left(\frac{dC_L}{d\alpha}\right)_w} \eta \frac{S_t}{S} \qquad (5a)$$

$$K_3 = -K_2 \frac{d\epsilon_1}{d\alpha} \qquad (5b)$$

$$K_5 = K_4 \frac{\left(\frac{dC_L}{d\alpha}\right)_t}{\left(\frac{dC_L}{d\alpha}\right)_w} \eta \frac{S_t}{S} \qquad (5c)$$

$$K_6 = -K_5 \frac{d\epsilon_1}{d\alpha} \qquad (5d)$$

$$K_8 = K_7 \frac{\left(\frac{dC_L}{d\alpha}\right)_t}{\left(\frac{dC_L}{d\alpha}\right)_w} \eta \frac{S_t}{S} \qquad (5e)$$

$$K_9 = -K_8 \frac{d\epsilon_1}{d\alpha} \tag{5f}$$

$$K_{11} = \left(\frac{dC_L}{d\alpha}\right)_t \eta \frac{S_t}{W} \frac{\rho}{2} V^2 \frac{l_t}{c} \tag{5g}$$

$$K_{12} = -\left(\frac{dC_L}{d\alpha}\right)_t \eta \frac{S_t}{W} \frac{\rho}{2} V^2 \frac{l_w}{c} \frac{d\epsilon_1}{d\alpha} \tag{5h}$$

$$K_{13} = \left(\frac{dC_L}{d\alpha}\right)_t \frac{\rho}{2} \frac{S_t}{W} V^2 \frac{d\alpha_e}{d\delta} \tag{5i}$$

$$K_{14} = \frac{\rho}{2} \frac{S_t}{W} V^2 \frac{c_t}{c} \frac{A_t}{A_t + 2} \left[\left(\frac{\partial C_L}{\partial D\delta}\right)_A + \left(\frac{dC_L}{d\alpha}\right)_t \left(\frac{\partial C_L}{\partial D\delta}\right)_B\right] \tag{5j}$$

Either experimental or theoretical values may be used for $\frac{d\alpha_e}{d\delta}$ appearing in equation (5i) and $\left(\frac{\partial C_L}{\partial D\delta}\right)_A$ and $\left(\frac{\partial C_L}{\partial D\delta}\right)_B$ in equation (5j). Some theoretical values are given in figure 1(a) of reference 3. Other expressions appearing in equations (5) and (5a) to (5j) are derived in the section entitled "Aerodynamic Coefficients." With the load-factor increments Δn_W and Δn_t expressed as a function of the aerodynamic coefficients and airplane configuration, equation (2) can be expressed in terms of the airplane parameters. All body moments are taken about the center of gravity and assumed positive in the nose-up direction.

The hinge moment due to the aerodynamic forces, assumed positive when the trailing edge is depressed, is set equal to the inertia hinge moment taken about the axis of rotation of the elevator. An expression for the total hinge moment H in terms of the aerodynamic characteristics and configuration of the airplane is required. The total hinge moment is made up of the aerodynamic hinge moment due to a change in angle of attack on the tail and the hinge moment due to elevator deflection.

The equation for the total hinge moment is as follows:

$$H = -K_{15} \frac{U}{U_{max}} + K_{16} \int_0^{s_1} \Delta n \; ds - K_{17}\theta + K_{18} D\theta +$$

$$K_{19} \int_0^{s_1} \epsilon_\alpha(s_1 - s)\Delta n(s) \; ds - K_{20} \int_0^{s_1} \epsilon_\alpha(s_1 - s)D\theta \; ds +$$

$$K_{21} \int_0^{s_1} \epsilon_\alpha(s_1 - s)D^2\theta \; ds - K_{22} \int_0^{s_1} \epsilon_g(s_1 - s)D \frac{U}{U_{max}} \; ds +$$

$$K_{23}\delta - K_{24} D\delta \tag{6}$$

where the coefficients are

$$K_{15} = -\frac{\rho}{2} U_{max}VS_eC_{h_\alpha}c_e \tag{6a}$$

$$K_{16} = -\frac{\rho}{2} cS_ec_egC_{h_\alpha} \tag{6b}$$

$$K_{17} = -\frac{\rho}{2} V^2S_ec_eC_{h_\alpha} \tag{6c}$$

$$K_{18} = \frac{\rho}{2} V^2S_ec_eC_{h_\alpha} \frac{l_t}{c} \tag{6d}$$

$$K_{19} = \frac{\rho}{2} cS_ec_eC_{h_\alpha} \frac{d\epsilon}{d\alpha} g \tag{6e}$$

$$K_{20} = \frac{\rho}{2} V^2S_ec_eC_{h_\alpha} \frac{d\epsilon}{d\alpha} \tag{6f}$$

$$K_{21} = -\frac{\rho}{2} V^2S_ec_eC_{h_\alpha} \frac{d\epsilon}{d\alpha} \frac{l_w}{c} \tag{6g}$$

$$K_{22} = \frac{\rho}{2} U_{max}VS_ec_eC_{h_\alpha} \frac{d\epsilon}{d\alpha} \tag{6h}$$

$$K_{23} = \frac{\rho}{2} V^2 S_e c_e C_{h_\delta} \tag{6i}$$

$$K_{24} = \frac{\rho}{2} V^2 S_e c_e \left\{ \frac{A_t}{A_t + 2} \left[\left(\frac{\partial C_h}{\partial D\delta}\right)_A + \left(\frac{dC_L}{d\alpha}\right)_t \left(\frac{\partial C_h}{\partial D\delta}\right)_B \right] \frac{c_t}{c} + \frac{4Q^2 K}{\rho V S_e c_e c} \right\} \tag{6j}$$

The Duhamel's integrals appearing in equation (6) are not due to any unsteady hinge moments, but merely afford a means of expressing the effects due to downwash. These integrals establish a relationship between the changes of angle of attack on the tail caused by the changes of angle of attack on the wing. If the value of the expression in brackets in equation (6j) cannot be determined experimentally, its value is given theoretically with the aid of figure 1(b) of reference 3.

For the purpose of simplifying the differential equations, equations (2) and (3) are rewritten as follows:

$$\left.\begin{aligned} D^2\theta &= K_{25}\,\Delta n_w - K_{26}\,\Delta n_t \\ K_{27}\,D^2\theta + K_{28}\,D^2\delta + K_{29}\,\Delta n &= H \end{aligned}\right\} \tag{7}$$

where the coefficients are

$$K_{25} = \frac{W l_w c^2}{I V^2} \tag{7a}$$

$$K_{26} = \frac{W l_t c^2}{I V^2} \tag{7b}$$

$$K_{27} = \left(I_e + H_e l_h - r I_s\right)\frac{V^2}{c^2} \tag{7c}$$

$$K_{28} = \left(I_e + r^2 I_s\right)\frac{V^2}{c^2} \tag{7d}$$

$$K_{29} = -\left(H_e + r H_s\right) g \tag{7e}$$

Aerodynamic Coefficients

Unsteady lift.- Theoretical unsteady-lift functions due to both a unit jump of angle of attack and a penetration of a sharp-edge gust are given in reference 1 for elliptical surfaces of several aspect ratios for incompressible flow. The normalized functions are reproduced here in terms of the whole reference chord of the wing as distinguished from the half-chord notation used in reference 1.

For a unit jump of angle of attack,

$$\left.\begin{aligned} C_{L_\alpha}(s)_{A=3} &= 1.000 - 0.283e^{-1.080s} \\ C_{L_\alpha}(s)_{A=6} &= 1.000 - 0.361e^{-0.762s} \\ C_{L_\alpha}(s)_{A=\infty} &= 1.000 - 0.165e^{-0.090s} - 0.335e^{-0.600s} \end{aligned}\right\} \qquad (8)$$

For a penetration of a sharp-edge gust,

$$\left.\begin{aligned} C_{L_g}(s)_{A=3} &= 1.000 - 0.679e^{-1.116s} - 0.227e^{-6.40s} \\ C_{L_g}(s)_{A=6} &= 1.000 - 0.448e^{-0.580s} - 0.272e^{-1.450s} - 0.193e^{-6.00s} \\ C_{L_g}(s)_{A=\infty} &= 1.000 - 0.236e^{-0.116s} - 0.513e^{-0.728s} - 0.171e^{-4.84s} \end{aligned}\right\} \qquad (9)$$

The expressions for the finite-aspect-ratio unsteady-lift functions were derived using the midchord of the elliptical surface as the reference chord.

In establishing the unsteady-lift functions on the tail, the exponents of the foregoing expressions must be multiplied by the ratio c/c_t in order to convert these expressions from tail chords to wing chords since they involve the distance traveled by the airplane measured in terms of the wing chord.

Unsteady downwash.- The theoretical value of the unsteady-downwash angle at the horizontal tail due to a unit jump of angle of attack on the wing is obtained by evaluating equation (1) of reference 4 for an average configuration. The result may be normalized and approximated by step functions to give the desired function ϵ_α. However, the curve in figure 2(a) is seen to be principally a function of the distance between the trailing edge of the wing and the leading edge of the tail and this distance, due to configuration plus an additional aerodynamic lag, accounts for the interval between negative and positive step functions. For the unsteady-downwash-angle function due to penetration of a gust ϵ_g, an additional lag of 0.75 chord is applied to these step functions. (See fig. 2(b).) Figure 9 of reference 4 indicates that tail length does not affect the magnitude of the function appreciably but merely tends to affect the interval of discontinuity. In order to generalize the functions ϵ_α and ϵ_g with respect to airplane-tail length, the lag interval is considered as a sum of the distance from the trailing edge of the wing to the leading edge of the tail plus an additional aerodynamic lag of 0.23 chord. The steady-state values of ϵ_α and ϵ_g were found to be approximately equal to the value for $d\epsilon/d\alpha$.

The effective unsteady-lift function on the tail due to the downwash from a unit jump of angle of attack on the wing may be determined by a similar procedure. With the results obtained from the solution of equation (1) of reference 4, equation (2) in this same reference can be evaluated. The results for an average configuration, when normalized and approximated by step functions in a manner referred to previously, give the desired function $C_{L_{\epsilon_\alpha}}$. (See fig. 3(a).) The effective unsteady-lift function due to downwash which results from the penetration of the wing in a sharp-edge gust $C_{L_{\epsilon_g}}$ is obtained by applying an additional lag of 0.75 chord to the step functions describing $C_{L_{\epsilon_\alpha}}$. (See fig. 3(b).) By an analysis similar to the one used to generalize the functions for unsteady-downwash angles, these two unsteady-lift functions are generalized with respect to tail length by considering the total-lag interval from negative to positive step functions as a sum of the distance from the trailing edge of the wing to the leading edge of the tail plus an additional aerodynamic lag of 0.80 chord. The steady-state value $d\epsilon_1/d\alpha$ is approximately equal to 90 percent of the value for $d\epsilon/d\alpha$.

Transformation to Matrices

Stick-free stability.- The integral relationships required for transforming the differential equation into a form for numerical solution are given in appendix A. In the process of obtaining the numerical solution, which is in the form of a recurrence formula, a number of constants were used to simplify the writing of the final solution. These constants, which appear as elements of a square matrix and several rectangular matrices, are in terms of the airplane stability coefficients K_1 to K_{29}, the transformed unsteady-lift functions B and C, and the increment Δs. The constants K_1 to K_{29} are defined in appendix B. The constants given in appendix B are for a six-point solution, that is, for a recurrence formula containing six previously known values of the variables to be solved. A six-point solution was selected in this paper since it is believed that the transformed unsteady-lift functions B and C can be approximated by six values at 1.5-chord intervals with sufficient accuracy even though only a moderate expenditure of time is required. The accuracy of the numerical solution would largely depend upon the size of the interval and the rapidity of the motions that the airplane undergoes when disturbed from equilibrium. For airplanes with a high mass parameter, a high moment of inertia about the center of gravity, and a high moment of inertia for the elevator and stick, the accuracy of the method is expected to be improved for the given six-point solution.

Certain terms appearing in the equations for the E and F constants defined in appendix B have the subscript $m - l_{C_L}$ and certain terms in the equations for the G constants in this same appendix have the subscript $m - l_\epsilon$, where $l_{C_L} = \dfrac{\frac{l_{wt}}{c} + 0.80}{\Delta s}$ and $l_\epsilon = \dfrac{\frac{l_{wt}}{c} + 0.23}{\Delta s}$. The physical significance of these terms in the equations for the constants is attributed to a lag of tail forces and hinge moments existing on the tail which were caused by the effects of the unsteady-downwash functions. (See figs. 2(a) and 3(a).) In order to express these effects mathematically, two values of the constant must be evaluated, one value with the term containing the subscript neglected and the other value with this term retained in its calculation. For values of $m < l_{C_L}$ and $m < l_\epsilon$, the first values of the constants are used in the calculations; for extended values of m, that is, $m \geqq l_{C_L}$ and $m \geqq l_\epsilon$, the second value is used. Since the numerical evaluation is made for a predetermined interval, the amount of lag specified by the discrepancies in the unsteady-downwash functions cannot conveniently be taken into account exactly; therefore, the amount of lag is taken to the nearest value of m.

When the constants E, F, and G are assembled into a square matrix and a series of rectangular matrices, the three transformed equations may be solved simultaneously by an inversion of the square matrix. The variables describing the motion of the airplane can then be solved for the stick-free condition by the following recurrence formula:

$$\begin{Bmatrix}\Delta n\\ D^2\theta\\ D^2\delta\end{Bmatrix}_m = [M]\left(\begin{Bmatrix}U_1\\ U_2\\ U_3\end{Bmatrix}_m + [S_1]\begin{Bmatrix}\Delta n\\ D^2\theta\\ D^2\delta\\ {}_I\Delta n\\ {}_ID^2\theta\\ {}_ID^2\delta\\ {}_{II}D^2\theta\\ {}_{II}D^2\delta\end{Bmatrix}_{m-1} + [S_2]\begin{Bmatrix}\Delta n\\ D^2\theta\\ D^2\delta\\ {}_ID^2\theta\\ {}_ID^2\delta\end{Bmatrix}_{m-2} + \ldots + [S_6]\begin{Bmatrix}\Delta n\\ D^2\theta\\ D^2\delta\\ {}_ID^2\theta\\ {}_ID^2\delta\end{Bmatrix}_{m-6}\right) \quad (10)$$

where the forcing-function matrix $\begin{Bmatrix}U_1\\ U_2\\ U_3\end{Bmatrix}$ is defined in appendix C for either a sharp-edge gust or one of arbitrary shape. With the aid of equations derived in appendix A, equation (10) is evaluated in a manner analogous to the solution of the wing penetrating a sharp-edge gust described in the section of the paper entitled "Solution of One Degree of Freedom."

The rectangular matrices $[S]$ are evaluated together with the square matrix $[M]$ in terms of the constants E, F, and G defined in appendix B. The square matrix represents the inverse of the matrix of the coefficients of Δn, $D^2\theta$, and $D^2\delta$ at the point m.

Stick-fixed stability.- The motions of the airplane with the elevator in a locked position may be determined from the stick-free stability equations by modifying the matrices appearing in equation (10). The effect of eliminating the elevator motions is to eliminate in the numerical solution certain E and F constants and all the G constants. Also, appropriate changes in the recurrence formula are made by modifying the square and rectangular matrices and the column matrices containing the variables and their integrations describing the elevator motions. In appendix B the modified values of the matrices $[M]_0$ and $[S]_0$ are presented and are written with a subscript zero to differentiate them from the stick-free condition.

The forcing functions U_1 and U_2 remain unchanged whereas the function U_3 is eliminated. In accordance with the previously cited changes, the following recurrence formula is obtained and can be evaluated in a manner similar to that for the sick-free condition:

$$\begin{Bmatrix}\Delta n\\ D^2\theta\end{Bmatrix}_m = [M]_0\left(\begin{Bmatrix}U_1\\ U_2\end{Bmatrix}_m + [S_1]_0\begin{Bmatrix}\Delta n\\ D^2\theta\\ {}_I\Delta n\\ {}_ID^2\theta\\ {}_{II}D^2\theta\end{Bmatrix}_{m-1} + [S_2]_0\begin{Bmatrix}\Delta n\\ D^2\theta\\ {}_ID^2\theta\end{Bmatrix}_{m-2} + \ldots + [S_6]_0\begin{Bmatrix}\Delta n\\ D^2\theta\\ {}_ID^2\theta\end{Bmatrix}_{m-6}\right) \quad (11)$$

APPLICATION OF METHOD

Solution of One Degree of Freedom

The comparison of a response to a dynamic system calculated exactly to the response calculated by the numerical method derived in appendix A will help clarify this method as well as give an estimate of the accuracy to be expected when applied to a more complicated system. In order to demonstrate the validity of the numerical integration method, a response to penetration of a sharp-edge gust has been calculated for a wing restrained in all but the vertical direction. The change in lift coefficient due to the gust and due to motion has been calculated as a function of wing penetration. Assuming a unit jump in angle of attack due to the gust $\frac{U}{V} = 1$ gives the equation of motion as

$$C_L(s_1) = \frac{dC_L}{d\alpha} C_{L_g}(s) - \frac{1}{\mu_g}\int_0^{s_1} C_{L_\alpha}(s_1 - s)C_L(s)\,ds \qquad (12)$$

where μ_g is defined as the mass parameter of the wing as follows:

$$\mu_g = \frac{2W}{g\frac{dC_L}{d\alpha}S\rho c}$$

In reference 1, Jones has solved equation (12) by operational methods for different values of mass parameter μ_g. All calculations were made for a wing of aspect ratio 6. In reference 1 the unsteady-lift functions C_{L_g} and C_{L_α} are defined as functions of half-chord while in this paper they are a function of whole chords as well as being expressed as a fraction of the steady-state value. The exact solution of the response using the unsteady-lift functions for aspect ratio 6 and a value of $\mu_g = 13.2$ (or in Jones' notation $\mu = 124$) is shown in figure 12 of reference 1. This curve is reproduced herein in figure 4 as a function of whole chords.

The transformation of equation (12) to the following recurrence formula can be accomplished with the aid of the equations found in appendix A:

$$C_{L_m} = \frac{dC_L}{d\alpha} C_{L_{g_m}} - \frac{1}{\mu_g} I_m^{C_L} + \frac{\Delta s}{\mu_g}\left[\frac{B_0}{3}, \frac{4B_1}{3}, \frac{2B_2}{3}, \frac{4B_3}{3}, \frac{2B_4}{3}, \frac{4B_5}{3}, \frac{B_6}{3}\right]\begin{Bmatrix} C_{L_m} \\ C_{L_{m-1}} \\ C_{L_{m-2}} \\ C_{L_{m-3}} \\ C_{L_{m-4}} \\ C_{L_{m-5}} \\ C_{L_{m-6}} \end{Bmatrix} \tag{13}$$

where

$$I_m^{C_L} = I_{m-1}^{C_L} + \Delta s\left(\frac{9}{24} C_{L_m} + \frac{19}{24} C_{L_{m-1}} - \frac{5}{24} C_{L_{m-2}} + \frac{1}{24} C_{L_{m-3}}\right) \tag{14}$$

The following numerical values are substituted in equation (13):

$$\frac{dC_L}{d\alpha} = 1.50\pi$$

$$\frac{1}{\mu_g} = 0.0760$$

$$\Delta s = 0.75 \text{ chord}$$

$$B(s) = 1 - C_{L_\alpha}(s)_{A=6}$$

(where the values $B_0, B_1, \ldots B_6$ are computed for the interval $\Delta s = 0.75$ chord from normalized values of the function $C_{L_\alpha}(s)_{A=6}$ shown in fig. 9 of reference 1). The reason for choosing a smaller interval is that the mass parameter μ_g is very low. The resulting recurrence formula is given as follows:

$$C_{L_m} = \frac{1}{1.014}\Big(1.50\pi C_{L_{g_m}} - 0.07576 I_{m-1}^{C_L} - 0.0299 C_{L_{m-1}} + 0.0162 C_{L_{m-2}} +$$

$$0.00256 C_{L_{m-3}} + 0.00140 C_{L_{m-4}} + 0.00167 C_{L_{m-5}} + 0.000265 C_{L_{m-6}}\Big) \tag{15}$$

The forcing function $C_{L_{g_m}}$ is computed for an interval of $\Delta s = 0.75$ chord from normalized values of the curve of figure 9 of reference 1. Equation (15) can be computed step by step in a manner similar to that described previously. The results of these numerical computations are shown in figure 4 together with the exact solution obtained by operational methods.

Solution of Stick-Fixed Condition

In order to establish a general procedure for performing the necessary calculations in a stability analysis, an illustrative example is presented in detail. The aerodynamic characteristics corresponding to the configuration of a modern transport airplane were selected. The parameters necessary for a stick-fixed analysis are shown in table I. Values of l_t and l_w are given for three center-of-gravity positions. Accordingly, the stability constants K_1 to K_{29} may be evaluated by equations (4a) to (4d), (5a) to (5j), (6a) to (6j), and (7a) to (7e). The results are shown in table II for those constants involving only the stick-fixed condition.

The unsteady-lift functions due to a unit jump of angle of attack on the wing or tail were calculated from equation (8). Note that the tail unsteady-lift functions must be expressed in wing chords by multiplying the value of s by c/c_t. The $B(s)$ and $C(s)$ functions for infinite aspect ratio for an interval of $\Delta s = 1.5$ chords is shown in table III. Together with the constants K_1 to K_{29}, $B(s)$, $C(s)$, and Δs, the E, F, and G elements can be calculated with the use of the equations in appendix B. These elements of the matrices $[M]$ and $[S]$ are shown both for the stick-free condition and for the stick-fixed condition in this appendix. Accordingly, the values for the constants E and F required for the stick-fixed condition are shown in table IV for an airplane with the center of gravity at 25 percent of the wing mean aerodynamic chord. The E and F elements are given as a function of the variable m in the recurrence formula. None of the G elements are required for the stick-fixed condition. Since a discontinuity due to the effective unsteady-lift coefficients on the tail caused by a unit jump of angle of attack on the wing $C_{L_{\epsilon_\alpha}}$ occurs at $m = 2$, two values of the elements E_1, F_1, and F_2 appear in the matrix $[M]_0$. Consequently, two values of the matrix $[M]_0$ have to be calculated.

The forcing functions represented by the column matrix $\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$ can be calculated by equation (C1) for a sharp-edge gust or by equations (C2) and (C3) for an arbitrary-shape gust. The unsteady-lift functions due to penetration of a sharp-edge gust $C_{L_g}(s)$ were calculated by equations (9). By a method similar to the one used for the unsteady-lift functions due to a unit jump of angle of attack on the wing, the tail unsteady-lift function due to penetration of a sharp-edge gust $C_{L_{gt}}(s)$ is expressed in wing chords by multiplying the value of s by c/c_t. The effective normalized unsteady-lift coefficient on the tail due to wing penetration $C_{L_{\epsilon_g}}(s)$ is shown in figure 3(b). Although the discontinuity in this function for the configuration considered occurs at $m = 3$ to the nearest integer, the value at $m = 2$ was chosen for simplification since the length of the discontinuity determined by reference 4 is not very accurate and the length of discontinuity due to lag of tail penetration $\tau = \dfrac{l_{wt} + c}{c\,\Delta s}$ is equal to 2 to the nearest integer. Therefore, all discontinuities are made to occur at $m = 2$ in the sample calculations. The forcing functions U_1 and U_2 have been calculated for a sharp-edge gust for an airplane with the center of gravity at 25 percent of the mean aerodynamic chord and the results are shown in table V.

The responses were calculated by equation (11) together with the formulas given by equations (A5) and (A8) of appendix A in a manner previously outlined for the calculation of the simplified example. In figure 5 the time histories of acceleration increment and angle of pitch about the center of gravity of the airplane penetrating a sharp-edge gust are shown from an evaluation of the recurrence formula for an average center-of-gravity position. The load-factor increment at the center of gravity can be broken up into wing and tail load-factor increments by evaluating equation (4) for determining $\Delta n_w(s)$ and then subtracting that function from $\Delta n(s)$ to obtain the load-factor increment on the tail $\Delta n_t(s)$. The results of these calculations are also shown in figure 5.

Although sample calculations are not given for the two other center-of-gravity positions, the results are shown in the form of maximum load-factor increments Δn_{max} for a sharp-edge gust and two gradient flat-topped gusts at three center-of-gravity positions in figure 6. The responses to the gradient gusts can either be calculated by equations (C2) and (C3) or by the alternate method previously described, which in this case would be the more expeditious since three gust shapes are considered in this paper.

DISCUSSION

An estimate of the accuracy to be expected in the solution of the equations of motion by the matrix method may be determined by an inspection of figure 4 where both the exact and numerical solutions are given for the lift coefficient of a wing penetrating a sharp-edge gust. The numerical solution approximates the exact solution very well, except for the discrepancies occurring at the beginning of the curve. This condition will exist in most cases because the first point of the motion is computed from only those terms at time m of the recurrence equation, the second point is computed from those terms at times m and $m - 1$, and the third point is computed from those terms at times $m, m - 1$, and $m - 2$. In like manner, the other points can be computed until the full recurrence equation is in use. The error reduces as the mass parameter increases since the motion for the first few calculated values is reduced. Consequently, for low mass parameters where the motion is rapid a smaller increment of Δs than 1.5 chords must be taken. If smaller increments are required, the values of $B(s)$ and $C(s)$ may have to be approximated by more than just six points, otherwise the assumption that these functions are zero after the last point is not justified. In order to overcome this difficulty, the recurrence equation should be calculated from $m = 0$ to $m = 6$ for an increment of Δs less than 1.5 chords. The recurrence formula is then reevaluated for an increment of $\Delta s = 1.5$ chords by using values of the variables and their corresponding integrals taken from the first calculation. In addition to the effects of mass parameter, the accuracy is also a function of the forcing function and their corresponding integrals taken from the first calculation at intervals of $\Delta s = 1.5$ chords. For forcing functions having a large initial slope, the motion for the first few steps would again be large and therefore the errors corresponding to its calculation would be substantially increased. As before, the errors may be minimized by using a larger number of small increments of Δs. Attention to the details of calculation for the sources of error pointed out, however, should lead to accuracies of calculation by the matrix method for the stick-fixed and stick-free conditions that are at least as good as those shown for the single-degree-of-freedom case in figure 4.

The assumptions made in the derivation of the differential equations impose certain limitations other than that of confining the treatment to longitudinal-stability effects alone. The length of the transient response to be calculated is limited since the effect of variations in forward speed was neglected. The inclusion of variations in forward speed, however, would unduly complicate the problem at the present time. Another limitation to consider is the effect the pilot has on the handling of the airplane. If the pilot's reactions are proportional either to the accelerations, velocities, rate of change of acceleration, or to various combinations of them, the restraint can be included in the calculation of the airplane motions by assuming these effects on the stick are due to an automatic pilot. However, knowledge of the pilot's behavior when flying through turbulent air is insufficient at the present time to warrant any such representation.

Examination of the sample total-load-factor response curve in figure 5 shows a discontinuity occurring approximately at the time when the tail first enters the gust. As seen, the discontinuity arises from the load calculated to be on the tail and is a result of the combination of the downwash effects on the tail with the effects of gust penetration. Although this discontinuity in the response does not actually exist, it is allowed to remain in this form for the determination of the response to arbitrary gust shapes when the response to the sharp-edge gust is used as an indicial response. In this way, the new response does not depend upon the manner in which the discontinuity is faired.

The curves shown in figure 6 illustrate the results of the use of the method for determining the effect of center-of-gravity position and gust-gradient distance on the gust-load response of a particular airplane. The curves show that the effect of airplane stability may be important in determining the gust load factor on an airplane. The gust-load-factor increment can be appreciably affected by the center-of-gravity position when the airplane is traversing large gradient gusts. A rearward center-of-gravity position, representing a decrease in stability, increases the maximum load-factor increment Δn_{max}. On the other hand, with the center of gravity in a forward position, the stability is increased and the corresponding maximum load-factor increment is decreased.

All the computations involved in the solution of the complicated integral equations of motion of an airplane penetrating a gust have been reduced by the method of this paper to simple, but lengthy, arithmetic operations. The use of some type of high-speed automatic computing machine (such as the Bell Telephone Laboratories X-66744 relay computer in use at the Langley Laboratory) would help overcome the difficulty of performing these computations. If the problem is set up in a machine, the length of time required for the computation of the stick-fixed stability equations for a single condition would usually require somewhat less than 1 hour to approximately 8 hours, depending on the type of machine. The initial time required for setting up the problem on the machine would appear less significant when the number of solutions of the equations is increased.

CONCLUDING REMARKS

The numerical method derived in matrix notation for solving the differential equations of motion provides a means for determining the effect of the various stability parameters on the gust load factor in a reasonable time considering the complexity of the problem. Although the method predicts the total load-factor increment on the airplane, separate wing and tail loads may be computed with comparative ease.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., November 8, 1949

APPENDIX A

SOLUTION OF EQUATIONS BY A METHOD OF NUMERICAL INTEGRATIONS

The solution of the integral equations may be accomplished by the calculus of finite differences. A recurrence formula is obtained in this manner for the variables in terms of their preceding values and the integrals of the preceding values.

Two mathematical operations are required for a solution of the equations by the method considered in this paper. First, a numerical evaluation of the various Duhamel's integrals expressed as functions of the variables describing the motions of the airplane is required. Second, a method is necessary to evaluate numerically a relationship between the variables and their respective first and second derivatives.

Since the indicial response $C_{L_\alpha}(s)$ appearing in the Duhamel's integrals approaches unity asymptotically, a convenient substitution can be employed so that the required integrations for succeeding intervals need not be continued indefinitely. For this simplification the substitutions required for the unsteady-lift functions are:

$$\left.\begin{aligned} B(s) &= 1 - C_{L_{\alpha_w}}(s) \\ C(s) &= 1 - C_{L_{\alpha_t}}(s) \end{aligned}\right\} \qquad \text{(A1)}$$

The functions $B(s)$ and $C(s)$ approach zero quite rapidly; therefore, the integration of the product of this function with the derivative of the arbitrary function may be carried out in a few steps. If the product of the functions to be integrated be $y(s)$, the desired value of the integral at point m can be evaluated in recurrence form for six intervals with the aid of Simpson's rule by the following equation:

$$\int_0^m y(s)\, ds = I_m^y = \Delta s\left[\frac{1}{3},\frac{4}{3},\frac{2}{3},\frac{4}{3},\frac{2}{3},\frac{4}{3},\frac{1}{3}\right]\begin{Bmatrix} y_m \\ y_{m-1} \\ y_{m-2} \\ y_{m-3} \\ y_{m-4} \\ y_{m-5} \\ y_{m-6} \end{Bmatrix} \qquad \text{(A2)}$$

where Δs is the interval of integration of the function $y(s)$.

The Duhamel's integrals containing the downwash functions may be simplified to ordinary type of integrals, because step functions represent the indicial responses. The integrals containing these functions may be replaced by two simple integrals, differing only in the upper limits of integration by an amount specified in the discontinuity occurring in the indicial responses. As an example, the following Duhamel's integral may be replaced by two integrals (see fig. (3)):

$$\int_0^{s_1} C_{L_{\epsilon_\alpha}}(s_1 - s)\Delta n(s)\, ds = -0.158\int_0^{s_1} \Delta n(s)\, ds + 1.158\int_0^{s_1-\left(\frac{l_{wt}}{c}+0.80\right)} \Delta n(s)\, ds \qquad (A3)$$

The evaluation of the Duhamel's integral now merely reduces to the second problem, namely that of relating the variable to its first derivative since the integration of the load factor is the first derivative of the displacement divided by the acceleration of gravity.

When approximating the successive derivatives of a function numerically, the accuracy of the operations decreases for each operation since the curve approximating each derivative would have to be of a lower order. As an example, if a function is replaced numerically by parabolic segments, the numerical approximations to the first derivative would be a series of straight-line segments; the second derivative, obviously, would be meaningless since its numerical approximation would be a constant for any value of the original function. On the other hand, a numerical method consisting of successive integrations will tend to reduce the error for each series of operations because the curve approximating each successive integration would be of a higher order. Thus, if a function is replaced by parabolic segments, the numerical approximation to the first integration would be a series of cubic segments and the numerical approximation to the second integration would be a series of quartic segments. Consequently, for each consecutive integration, the accuracy of the numerical approximations tends to increase.

In the numerical solution, the highest derivatives of the variables can be conveniently treated in the differential equations. Any derivatives of lower order may then be obtained by successive numerical integrations. In the previously derived equations of motion, no derivatives of higher order than the second occur; therefore, two successive integrations are required to establish the relationship between the variables and their respective derivatives.

The first numerical integration of the variable $y(s)$ may be performed in the following manner:

$$\int_0^{s_m} y(s)\,ds = \int_0^{s_{m-1}} y(s)\,ds + \int_{s_{m-1}}^{s_m} y(s)\,ds \tag{A4}$$

When the function and its integral from 0 to s_{m-1} are known, the value of the integral from the increment s_{m-1} to s_m may be obtained from the known values $y_{m-1}, y_{m-2}, y_{m-3}, \ldots$, and the unknown value y_m. The integration over the increment from s_{m-1} to s_m can be performed in a manner analogous to that used in Simpson's rule. In lieu of passing a parabola through these points, a cubic assumed to pass through four points may be used to accomplish the same purpose with increased accuracy. Equation (A4) may be rewritten

$$I_m^y = I_{m-1}^y + \Delta s\left\lfloor \frac{9}{24}, \frac{19}{24}, -\frac{5}{24}, \frac{1}{24} \right\rfloor \begin{Bmatrix} y_m \\ y_{m-1} \\ y_{m-2} \\ y_{m-3} \end{Bmatrix} \tag{A5}$$

For the double integration, an analysis similar to the one previously derived may be utilized

$$\int_0^{s_m}\int_0^{s} y(s)\,ds\,ds = \int_0^{s_m} I^y\,ds = \int_0^{s_{m-1}} I^y\,ds + \int_{s_{m-1}}^{s_m} I^y\,ds \tag{A6}$$

or, in terms of the double integration symbol II^y,

$$II_m^y = II_{m-1}^y + \Delta s\left\lfloor \frac{9}{24}, \frac{19}{24}, -\frac{5}{24}, \frac{1}{24} \right\rfloor \begin{Bmatrix} I_m^y \\ I_{m-1}^y \\ I_{m-2}^y \\ I_{m-3}^y \end{Bmatrix} \tag{A7}$$

If the value of I_m^y from equation (A5) is substituted in equation (A7), the double integration of the function $y(s)$ can be written in terms of the previously determined values of the function, its first and second integrations, and the function $y(s)$ at the next desired value s_m as follows:

$$II_m^y = II_{m-1}^y + (\Delta s)^2 \frac{9}{24}\left\lfloor \frac{9}{24}, \frac{19}{24}, -\frac{5}{24}, \frac{1}{24} \right\rfloor \begin{Bmatrix} y_m \\ y_{m-1} \\ y_{m-2} \\ y_{m-3} \end{Bmatrix} + \Delta s \left\lfloor \frac{7}{6}, -\frac{5}{24}, \frac{1}{24} \right\rfloor \begin{Bmatrix} I_{m-1}^y \\ I_{m-2}^y \\ I_{m-3}^y \end{Bmatrix} \qquad \text{(A8)}$$

The solution of the equations of motion by equations (A5) and (A8), together with the evaluation of Duhamel's integral with the aid of equation (A2), can now be accomplished by a systematic procedure. Complete histories of the accelerations, velocities, and displacements of the airplane are computed simply by evaluating, respectively, the variables y_m, I_m^y and II_m^y. The procedure can be continued for as many steps as desired.

In order to satisfy the initial conditions of the differential equations, the initial values of displacements, velocities, and accelerations may be substituted into the numerical solutions. For the problem considered in this paper, all the initial values of the displacements and velocities are zero, since the airplane is assumed to be in steady level flight before being disturbed by the gust. If the gust velocity also increases from a zero value at time zero the initial accelerations are zero as well. The numerical solution may be regarded as a recurrence formula for the values y_m, I_m^y, and II_m^y which constitute the acceleration, velocity, and displacement, respectively. With the values at m together with those found at $m - 1$, $m - 2$, $m - 3$, . . . , the values at $m + 1$ can be found. In like manner, the other values of m can be obtained.

APPENDIX B

CONSTANTS USED IN CALCULATIONS

When the differential equations of motion are transformed for numerical evaluation, they can be written more simply by combining groups of constants due to the airplane parameters into one constant. These substitutions are made in equations (4) to (7) and are tabulated as follows for convenience of calculation:

$$K_1 = \left(\frac{dC_L}{d\alpha}\right)_w \frac{\rho}{2} VS \frac{U_{max}}{W}$$

$$K_2 = K_1 \frac{\left(\frac{dC_L}{d\alpha}\right)_t}{\left(\frac{dC_L}{d\alpha}\right)_w} \eta \frac{S_t}{S}$$

$$K_3 = -K_2 \frac{d\epsilon_1}{d\alpha}$$

$$K_4 = \left(\frac{dC_L}{d\alpha}\right)_w \frac{\rho}{2} \frac{S}{W} cg$$

$$K_5 = K_4 \frac{\left(\frac{dC_L}{d\alpha}\right)_t}{\left(\frac{dC_L}{d\alpha}\right)_w} \eta \frac{S_t}{S}$$

$$K_6 = -K_5 \frac{d\epsilon_1}{d\alpha}$$

$$K_7 = \left(\frac{dC_L}{d\alpha}\right)_w \frac{S}{W} \frac{\rho}{2} V^2$$

$$K_8 = K_7 \frac{\left(\frac{dC_L}{d\alpha}\right)_t}{\left(\frac{dC_L}{d\alpha}\right)_w} \eta \frac{S_t}{S}$$

$$K_9 = -K_8 \frac{d\epsilon_1}{d\alpha}$$

$$K_{10} = \left(\frac{dC_L}{d\alpha}\right)_w \frac{S}{W} \frac{\rho}{2} V^2 \frac{l_w}{c}$$

$$K_{11} = \left(\frac{dC_L}{d\alpha}\right)_t \eta \frac{S_t}{W} \frac{\rho}{2} V^2 \frac{l_t}{c}$$

$$K_{12} = -\left(\frac{dC_L}{d\alpha}\right)_t \eta \frac{S_t}{W} \frac{\rho}{2} V^2 \frac{l_w}{c} \frac{d\epsilon_1}{d\alpha}$$

$$K_{13} = \left(\frac{dC_L}{d\alpha}\right)_t \frac{\rho}{2} \frac{S_t}{W} V^2 \frac{d\alpha_e}{d\delta}$$

$$K_{14} = \frac{\rho}{2} \frac{S_t}{W} V^2 \frac{c_t}{c} \frac{A_t}{A_t + 2} \left[\left(\frac{\partial C_L}{\partial D\delta}\right)_A + \left(\frac{dC_L}{d\alpha}\right)_t \left(\frac{\partial C_L}{\partial D\delta}\right)_B\right]$$

$$K_{15} = -\frac{\rho}{2} U_{max} V S_e C_{h_\alpha} c_e$$

$$K_{16} = -\frac{\rho}{2} c S_e c_e g C_{h_\alpha}$$

$$K_{17} = -\frac{\rho}{2} V^2 S_e c_e C_{h_\alpha}$$

$$K_{18} = \frac{\rho}{2} V^2 S_e c_e C_{h_\alpha} \frac{l_t}{c}$$

$$K_{19} = \frac{\rho}{2} c S_e c_e C_{h_\alpha} \frac{d\epsilon}{d\alpha} g$$

$$K_{20} = \frac{\rho}{2} V^2 S_e c_e C_{h_\alpha} \frac{d\epsilon}{d\alpha}$$

$$K_{21} = -\frac{\rho}{2} V^2 S_e c_e C_{h_\alpha} \frac{d\epsilon}{d\alpha} \frac{l_w}{c}$$

$$K_{22} = \frac{\rho}{2} U_{max} V S_e c_e C_{h_\alpha} \frac{d\epsilon}{d\alpha}$$

$$K_{23} = \frac{\rho}{2} V^2 S_e c_e C_{h_\delta}$$

$$K_{24} = \frac{\rho}{2} V^2 S_e c_e \left\{\frac{A_t}{A_t + 2} \left[\left(\frac{\partial C_h}{\partial D\delta}\right)_A + \left(\frac{dC_L}{d\alpha}\right)_t \left(\frac{\partial C_h}{\partial D\delta}\right)_B\right] \frac{c_t}{c} + \frac{4Q^2 K}{\rho V S_e c_e c}\right\}$$

$$K_{25} = \frac{W l_w c^2}{IV^2}$$

$$K_{26} = \frac{W l_t c^2}{IV^2}$$

$$K_{27} = \left(I_e + H_e l_h - rI_s\right) \frac{V^2}{c^2}$$

$$K_{28} = \left(I_e + r^2 I_s\right) \frac{V^2}{c^2}$$

$$K_{29} = -(H_e + rH_s)g$$

When the equations of motion are transformed to numerical form by making use of the integral equations derived in appendix A, terms common to the same time interval, that is, at times m, $m - 1$, $m - 2$, . . . , are grouped into one constant. These constants are obtained for each time interval, each equation of motion, and each variable and its integrals describing the motion of the airplane. The constants formed by this grouping procedure are elements in the square and rectangular matrices in the recurrence formula.

The constants are grouped with respect to the three equations of motion. The following constants are due to the transformation of equation (1) into recurrence form:

$$E_1 = 1 + \frac{9}{24} \Delta s \left[K_4 - 0.158K_6 + (1.158K_6)_{m-l_{C_L}} + K_5\right] - \frac{1}{3} \Delta s \left(K_4 B_0 + K_5 C_0\right)$$

$$E_2 = \left(\frac{9}{24} \Delta s\right)^2 \left[-K_7 - K_8 + 0.158K_9 - \left(1.158K_9\right)_{m-l_{C_L}}\right] + \frac{9}{24} \Delta s \left[K_7 \Delta s \frac{B_0}{3} + K_8 \Delta s \frac{C_0}{3} + K_{10} - K_{11} - 0.158K_{12} + \left(1.158K_{12}\right)_{m-l_{C_L}}\right] + \frac{1}{3} \Delta s \left(-K_{10} B_0 + K_{11} C_0\right)$$

$$E_3 = \frac{9}{24} \Delta s \left(-K_{13} \frac{9}{24} \Delta s + K_{13} \Delta s \frac{C_0}{3} - K_{14}\right) + K_{14} \Delta s \frac{C_0}{3}$$

$$E_4 = -K_4 - K_5 + 0.158K_6 - \left(1.158K_6\right)_{m-l_{C_L}-1}$$

$$E_5 = \frac{7}{6}\Delta s\left[K_7 + K_8 - 0.158K_9 + \left(1.158K_9\right)_{m-l_{C_L}-1}\right] - \frac{1}{3}\Delta s\left(K_7B_0 + 4K_7B_1 + K_8C_0 + 4K_8C_1\right) - K_{10} + K_{11} + 0.158K_{12} - \left(1.158K_{12}\right)_{m-l_{C_L}-1}$$

$$E_6 = \frac{5}{24}\Delta s\left[-K_7 - K_8 + 0.158K_9 - \left(1.158K_9\right)_{m-l_{C_L}-2}\right] - \frac{2}{3}\Delta s\left(K_7B_2 + K_8C_2\right)$$

$$E_7 = \frac{1}{24}\Delta s\left[K_7 + K_8 - 0.158K_9 + \left(1.158K_9\right)_{m-l_{C_L}-3}\right] - \frac{4}{3}\Delta s\left(K_7B_3 + K_8C_3\right)$$

$$E_8 = \frac{2}{3}\Delta s\left(-K_7B_4 - K_8C_4\right)$$

$$E_9 = \frac{4}{3}\Delta s\left(-K_7B_5 - K_8C_5\right)$$

$$E_{10} = \frac{1}{3}\Delta s\left(-K_7B_6 - K_8C_6\right)$$

$$E_{11} = K_7 + K_8 - 0.158K_9 + \left(1.158K_9\right)_{m-l_{C_L}-1}$$

$$E_{12} = K_{13}\Delta s\left(\frac{7}{6} - \frac{C_0}{3} - \frac{4}{3}C_1\right) + K_{14}$$

$$E_{13} = K_{13}\Delta s\left(-\frac{5}{24} - \frac{2}{3}C_2\right)$$

$$E_{14} = K_{13}\Delta s\left(\frac{1}{24} - \frac{4}{3}C_3\right)$$

$$E_{15} = -K_{13}\Delta s\,\frac{2}{3}C_4$$

$$E_{16} = -K_{13}\Delta s\,\frac{4}{3}C_5$$

$$E_{17} = -K_{13}\Delta s\,\frac{1}{3}C_6$$

$$E_{18} = \frac{19}{24}\Delta s\left[-K_4 - K_5 + 0.158K_6 - \left(1.158K_6\right)_{m-l_{C_L}-1}\right] + \frac{4}{3}\Delta s\left(K_4B_1 + K_5C_1\right)$$

$$E_{19} = \frac{5}{24}\Delta s\left[K_4 + K_5 - 0.158K_6 + \left(1.158K_6\right)_{m-l_{C_L}-2}\right] + \frac{2}{3}\Delta s\left(K_4B_2 + K_5C_2\right)$$

$$E_{20} = \frac{1}{24}\Delta s\left[-K_4 - K_5 + 0.158K_6 - \left(1.158K_6\right)_{m-l_{C_L}-3}\right] + \frac{4}{3}\Delta s\left(K_4B_3 + K_5C_3\right)$$

$$E_{21} = \frac{2}{3}\Delta s\left(K_4B_4 + K_5C_4\right)$$

$$E_{22} = \frac{4}{3}\Delta s\left(K_4B_5 + K_5C_5\right)$$

$$E_{23} = \frac{1}{3}\Delta s\left(K_4B_6 + K_5C_6\right)$$

$$E_{24} = \frac{19}{24}(\Delta s)^2\left[K_7\frac{9}{24} - K_7\frac{B_0}{3} + K_8\frac{9}{24} - K_8\frac{C_0}{3} - 0.158K_9\frac{9}{24} + \left(1.158K_9\frac{9}{24}\right)_{m-l_{C_L}-1}\right] + \Delta s\left[-K_{10}\frac{19}{24} + K_{10}\frac{4}{3}B_1 + K_{11}\frac{19}{24} - K_{11}\frac{4}{3}C_1 + 0.158K_{12}\frac{19}{24} - \left(1.158K_{12}\frac{19}{24}\right)_{m-l_{C_L}-1}\right]$$

$$E_{25} = \frac{5}{24}(\Delta s)^2\left[-K_7\frac{9}{24} + K_7\frac{B_0}{3} - K_8\frac{9}{24} + K_8\frac{C_0}{3} + 0.158K_9\frac{9}{24} - \left(1.158K_9\frac{9}{24}\right)_{m-l_{C_L}-2}\right] + \Delta s\left[K_{10}\frac{5}{24} + K_{10}\frac{2}{3}B_2 - K_{11}\frac{5}{24} - K_{11}\frac{2}{3}C_2 - 0.158K_{12}\frac{5}{24} + \left(1.158K_{12}\frac{5}{24}\right)_{m-l_{C_L}-2}\right]$$

$$E_{26} = \frac{1}{24}(\Delta s)^2\left[K_7\frac{9}{24} - K_7\frac{B_0}{3} + K_8\frac{9}{24} - K_8\frac{C_0}{3} - 0.158K_9\frac{9}{24} + \left(1.158K_9\frac{9}{24}\right)_{m-l_{C_L}-3}\right] + \Delta s\left[-K_{10}\frac{1}{24} + K_{10}\frac{4}{3}B_3 + K_{11}\frac{1}{24} - K_{11}\frac{4}{3}C_3 + 0.158K_{12}\frac{1}{24} - \left(1.158K_{12}\frac{1}{24}\right)_{m-l_{C_L}-3}\right]$$

$$E_{27} = \frac{2}{3}\Delta s\left(K_{10}B_4 - K_{11}C_4\right)$$

$$E_{28} = \frac{4}{3}\Delta s\left(K_{10}B_5 - K_{11}C_5\right)$$

$$E_{29} = \frac{1}{3} \Delta s \left(K_{10}B_6 - K_{11}C_6\right)$$

$$E_{30} = \frac{19}{24} \Delta s \left(K_{13} \frac{9}{24} \Delta s - K_{13} \Delta s \frac{C_0}{3} + K_{14}\right) - K_{14} \Delta s \frac{4}{3} C_1$$

$$E_{31} = \frac{5}{24} \Delta s \left(-K_{13} \Delta s \frac{9}{24} + K_{13} \Delta s \frac{C_0}{3} - K_{14}\right) - K_{14} \Delta s \frac{2}{3} C_2$$

$$E_{32} = \frac{1}{24} \Delta s \left(K_{13} \Delta s \frac{9}{24} - K_{13} \Delta s \frac{C_0}{3} + K_{14}\right) - K_{14} \Delta s \frac{4}{3} C_3$$

$$E_{33} = -K_{14} \Delta s \frac{2}{3} C_4$$

$$E_{34} = -K_{14} \Delta s \frac{4}{3} C_5$$

$$E_{35} = -K_{14} \Delta s \frac{1}{3} C_6$$

$$E_{36} = K_{13}$$

The following constants are due to the transformation of equation (2) into recurrence form:

$$F_1 = \frac{9}{24} \Delta s \left[K_4K_{25} - K_5K_{26} + 0.158K_6K_{26} - \left(1.158K_6K_{26}\right)_{m-l_{C_L}}\right] - \frac{1}{3} \Delta s \left(K_4K_{25}B_0 - K_5K_{26}C_0\right)$$

$$F_2 = 1 + \left(\frac{9}{24} \Delta s\right)^2 \left[-K_7K_{25} + K_8K_{26} - 0.158K_9K_{26} + \left(1.158K_9K_{26}\right)_{m-l_{C_L}}\right] + \frac{9}{24} \Delta s \left[K_7K_{25} \Delta s \frac{B_0}{3} + K_{10}K_{25} - K_8K_{26} \Delta s \frac{C_0}{3} + K_{11}K_{26} + 0.158K_{12}K_{26} - \left(1.158K_{12}K_{26}\right)_{m-l_{C_L}}\right] + \frac{1}{3} \Delta s \left(-K_{10}K_{25}B_0 - K_{11}K_{26}C_0\right)$$

$$F_3 = \frac{9}{24} \Delta s \left(K_{13}K_{26} \frac{9}{24} \Delta s - K_{13}K_{26} \Delta s \frac{C_0}{3} + K_{14}K_{26}\right) - K_{14}K_{26} \Delta s \frac{C_0}{3}$$

$$F_4 = -K_4K_{25} + K_5K_{26} - 0.158K_6K_{26} + \left(1.158K_6K_{26}\right)_{m-l_{C_L}-1}$$

$$F_5 = \frac{7}{6}\,\Delta s\left[K_7K_{25} - K_8K_{26} + 0.158K_9K_{26} - \left(1.158K_9K_{26}\right)_{m-l_{C_L}-1}\right] +$$

$$\frac{1}{3}\,\Delta s\left(-K_7K_{25}B_0 - 4K_7K_{25}B_1 + K_8K_{26}C_0 + 4K_8K_{26}C_1\right) - K_{10}K_{25} - K_{11}K_{26} -$$

$$0.158K_{12}K_{26} + \left(1.158K_{12}K_{26}\right)_{m-l_{C_L}-1}$$

$$F_6 = \frac{5}{24}\,\Delta s\left[-K_7K_{25} + K_8K_{26} - 0.158K_9K_{26} + \left(1.158K_9K_{26}\right)_{m-l_{C_L}-2}\right] +$$

$$\frac{2}{3}\,\Delta s\left(-K_7K_{25}B_2 + K_8K_{26}C_2\right)$$

$$F_7 = \frac{1}{24}\,\Delta s\left[K_7K_{25} - K_8K_{26} + 0.158K_9K_{26} - \left(1.158K_9K_{26}\right)_{m-l_{C_L}-3}\right] -$$

$$\frac{4}{3}\,\Delta s\left(K_7K_{25}B_3 - K_8K_{26}C_3\right)$$

$$F_8 = \frac{2}{3}\,\Delta s\left(-K_7K_{25}B_4 + K_8K_{26}C_4\right)$$

$$F_9 = \frac{4}{3}\,\Delta s\left(-K_7K_{25}B_5 + K_8K_{26}C_5\right)$$

$$F_{10} = \frac{1}{3}\,\Delta s\left(-K_7K_{25}B_6 + K_8K_{26}C_6\right)$$

$$F_{11} = K_7K_{25} - K_8K_{26} + 0.158K_9K_{26} - \left(1.158K_9K_{26}\right)_{m-l_{C_L}-1}$$

$$F_{12} = -K_{13}K_{26}\,\Delta s\,\frac{7}{6} + K_{13}K_{26}\,\Delta s\,\frac{C_0}{3} + K_{13}K_{26}\,\Delta s\,\frac{4}{3}\,C_1 - K_{14}K_{26}$$

$$F_{13} = \Delta s\left(K_{13}K_{26}\,\frac{5}{24} + K_{13}K_{26}\,\frac{2}{3}\,C_2\right)$$

$$F_{14} = \Delta s\left(-K_{13}K_{26}\,\frac{1}{24} + K_{13}K_{26}\,\frac{4}{3}\,C_3\right)$$

$$F_{15} = K_{13}K_{26}\,\Delta s\,\frac{2}{3}\,C_4$$

$$F_{16} = K_{13}K_{26}\,\Delta s\,\frac{4}{3}\,C_5$$

$$F_{17} = K_{13}K_{26}\,\Delta s\,\frac{1}{3}\,C_6$$

$$F_{18} = \frac{19}{24}\,\Delta s\left[-K_4K_{25} + K_5K_{26} - 0.158K_6K_{26} + \left(1.158K_6K_{26}\right)_{m-l_{C_L}-1}\right] + \frac{4}{3}\,\Delta s\left(K_4K_{25}B_1 - K_5K_{26}C_1\right)$$

$$F_{19} = \frac{5}{24}\,\Delta s\left[K_4K_{25} + 0.158K_6K_{26} - \left(1.158K_6K_{26}\right)_{m-l_{C_L}-2} - K_5K_{26}\right] + \frac{2}{3}\,\Delta s\left(K_4K_{25}B_2 - K_5K_{26}C_2\right)$$

$$F_{20} = \frac{1}{24}\,\Delta s\left[-K_4K_{25} + K_5K_{26} - 0.158K_6K_{26} + \left(1.158K_6K_{26}\right)_{m-l_{C_L}-3}\right] + \frac{4}{3}\,\Delta s\left(K_4K_{25}B_3 - K_5K_{26}C_3\right)$$

$$F_{21} = \frac{2}{3}\,\Delta s\left(K_4K_{25}B_4 - K_5K_{26}C_4\right)$$

$$F_{22} = \frac{4}{3}\,\Delta s\left(K_4K_{25}B_5 - K_5K_{26}C_5\right)$$

$$F_{23} = \frac{1}{3}\,\Delta s\left(K_4K_{25}B_6 - K_5K_{26}C_6\right)$$

$$F_{24} = \frac{19}{24}\,(\Delta s)^2\left[K_7K_{25}\,\frac{9}{24} - K_7K_{25}\,\frac{B_0}{3} - K_8K_{26}\,\frac{9}{24} + K_8K_{26}\,\frac{C_0}{3} + 0.158K_9K_{26}\,\frac{9}{24} - \left(1.158K_9K_{26}\,\frac{9}{24}\right)_{m-l_{C_L}-1}\right] + \Delta s\left[-K_{10}K_{25}\,\frac{19}{24} + K_{10}K_{25}\,\frac{4}{3}\,B_1 - K_{11}K_{26}\,\frac{19}{24} + K_{11}K_{26}\,\frac{4}{3}\,C_1 - 0.158K_{12}K_{26}\,\frac{19}{24} + \left(1.158K_{12}K_{26}\,\frac{19}{24}\right)_{m-l_{C_L}-1}\right]$$

$$F_{25} = \frac{5}{24}\,(\Delta s)^2\left[-K_7K_{25}\,\frac{9}{24} + K_7K_{25}\,\frac{B_0}{3} + K_8K_{26}\,\frac{9}{24} - K_8K_{26}\,\frac{C_0}{3} - 0.158K_9K_{26}\,\frac{9}{24} + \left(1.158K_9K_{26}\,\frac{9}{24}\right)_{m-l_{C_L}-2}\right] + \Delta s\left[K_{10}K_{25}\,\frac{5}{24} + K_{10}K_{25}\,\frac{2}{3}\,B_2 + K_{11}K_{26}\,\frac{5}{24} + K_{11}K_{26}\,\frac{2}{3}\,C_2 + 0.158K_{12}K_{26}\,\frac{5}{24} - \left(1.158K_{12}K_{26}\,\frac{5}{24}\right)_{m-l_{C_L}-2}\right]$$

$$F_{26} = \frac{1}{24}(\Delta s)^2\left[K_7K_{25}\frac{9}{24} - K_7K_{25}\frac{B_0}{3} - K_8K_{26}\frac{9}{24} + K_8K_{26}\frac{C_0}{3} + 0.158K_9K_{26}\frac{9}{24} - \left(1.158K_9K_{26}\frac{9}{24}\right)_{m-l_{C_L}-3}\right] + \Delta s\left[-K_{10}K_{25}\frac{1}{24} + K_{10}K_{25}\frac{4}{3}B_3 - K_{11}K_{26}\frac{1}{24} + K_{11}K_{26}\frac{4}{3}C_3 - 0.158K_{12}K_{26}\frac{1}{24} + \left(1.158K_{12}K_{26}\frac{1}{24}\right)_{m-l_{C_L}-3}\right]$$

$$F_{27} = \frac{2}{3}\Delta s\left(K_{10}K_{25}B_4 + K_{11}K_{26}C_4\right)$$

$$F_{28} = \frac{4}{3}\Delta s\left(K_{10}K_{25}B_5 + K_{11}K_{26}C_5\right)$$

$$F_{29} = \frac{1}{3}\Delta s\left(K_{10}K_{25}B_6 + K_{11}K_{26}C_6\right)$$

$$F_{30} = \frac{19}{24}\Delta s\left(-K_{13}K_{26}\Delta s\frac{9}{24} + K_{13}K_{26}\Delta s\frac{C_0}{3} - K_{14}K_{26}\right) + K_{14}K_{26}\Delta s\frac{4}{3}C_1$$

$$F_{31} = \frac{5}{24}\Delta s\left(K_{13}K_{26}\Delta s\frac{9}{24} - K_{13}K_{26}\Delta s\frac{C_0}{3} + K_{14}K_{26}\right) + K_{14}K_{26}\Delta s\frac{2}{3}C_2$$

$$F_{32} = \frac{1}{24}\Delta s\left(-K_{13}K_{26}\Delta s\frac{9}{24} + K_{13}K_{26}\Delta s\frac{C_0}{3} - K_{14}K_{26}\right) + K_{14}K_{26}\Delta s\frac{4}{3}C_3$$

$$F_{33} = K_{14}K_{26}\Delta s\frac{2}{3}C_4$$

$$F_{34} = K_{14}K_{26}\Delta s\frac{4}{3}C_5$$

$$F_{35} = K_{14}K_{26}\Delta s\frac{1}{3}C_6$$

$$F_{36} = -K_{13}K_{26}$$

The following constants are due to the transformation of equation (3) into recurrence form:

$$G_1 = K_{29} - \frac{9}{24}\Delta s\left[K_{16} - 0.167K_{19} + \left(1.167K_{19}\right)_{m-l_\epsilon}\right]$$

$$G_2 = K_{27} + \left(\frac{9}{24}\Delta s\right)^2 \left[K_{17} - 0.167K_{20} + \left(1.167K_{20}\right)_{m-l_\epsilon}\right] -$$

$$\frac{9}{24}\Delta s \left[K_{18} - 0.167K_{21} + \left(1.167K_{21}\right)_{m-l_\epsilon}\right]$$

$$G_3 = K_{28} - K_{23}\left(\Delta s\,\frac{9}{24}\right)^2 - K_{24}\,\Delta s\,\frac{9}{24}$$

$$G_4 = K_{16} - 0.167K_{19} + \left(1.167K_{19}\right)_{m-l_\epsilon-1}$$

$$G_5 = \frac{7}{6}\Delta s\left[-K_{17} + 0.167K_{20} - \left(1.167K_{20}\right)_{m-l_\epsilon-1}\right] + K_{18} - 0.167K_{21} +$$

$$\left(1.167K_{21}\right)_{m-l_\epsilon-1}$$

$$G_6 = \frac{5}{24}\Delta s\left[K_{17} - 0.167K_{20} + \left(1.167K_{20}\right)_{m-l_\epsilon-2}\right]$$

$$G_7 = \frac{1}{24}\Delta s\left[-K_{17} + 0.167K_{20} - \left(1.167K_{20}\right)_{m-l_\epsilon-3}\right]$$

$$G_8 = K_{23}\,\Delta s\,\frac{7}{6} + K_{24}$$

$$G_9 = -K_{23}\,\Delta s\,\frac{5}{24}$$

$$G_{10} = K_{23}\,\Delta s\,\frac{1}{24}$$

$$G_{11} = \frac{19}{24}\Delta s\left[K_{16} - 0.167K_{19} + \left(1.167K_{19}\right)_{m-l_\epsilon-1}\right]$$

$$G_{12} = \frac{5}{24}\Delta s\left[-K_{16} + 0.167K_{19} - \left(1.167K_{19}\right)_{m-l_\epsilon-2}\right]$$

$$G_{13} = \frac{1}{24}\Delta s\left[K_{16} - 0.167K_{19} + \left(1.167K_{19}\right)_{m-l_\epsilon-3}\right]$$

$$G_{14} = \frac{19}{24}\Delta s\left[-K_{17}\,\Delta s\,\frac{9}{24} + K_{18} + 0.167K_{20}\,\Delta s\,\frac{9}{24} - \left(1.167K_{20}\,\Delta s\,\frac{9}{24}\right)_{m-l_\epsilon-1} -\right.$$

$$\left.0.167K_{21} + \left(1.167K_{21}\right)_{m-l_\epsilon-1}\right]$$

$$G_{15} = \frac{5}{24} \Delta s \left[K_{17} \Delta s \frac{9}{24} - K_{18} - 0.167K_{20} \Delta s \frac{9}{24} + \left(1.167K_{20} \Delta s \frac{9}{24}\right)_{m-l_\epsilon-2} + 0.167K_{21} - \left(1.167K_{21}\right)_{m-l_\epsilon-2}\right]$$

$$G_{16} = \frac{1}{24} \Delta s \left[-K_{17} \Delta s \frac{9}{24} + K_{18} + 0.167K_{20} \Delta s \frac{9}{24} - \left(1.167K_{20} \Delta s \frac{9}{24}\right)_{m-l_\epsilon-2} - 0.167K_{21} + \left(1.167K_{21}\right)_{m-l_\epsilon-3}\right]$$

$$G_{17} = \frac{19}{24} \Delta s\left(K_{23} \Delta s \frac{9}{24} + K_{24}\right)$$

$$G_{18} = \frac{5}{24} \Delta s \left(-K_{23} \Delta s \frac{19}{24} - K_{24}\right)$$

$$G_{19} = \frac{1}{24} \Delta s\left(K_{23} \Delta s \frac{9}{24} + K_{24}\right)$$

$$G_{20} = K_{23}$$

$$G_{21} = -K_{17} + 0.167K_{20} - \left(1.167K_{20}\right)_{m-l_\epsilon-1}$$

The elements due to the constants E, F, and G are put into matrices which modify the variables and the integrals describing the motion of the airplane. The matrix describing the variables at time m is a square matrix that must be inverted in order that the variables at time m can be solved simultaneously. The elements for this inverted matrix form the square matrix $[M]$ which appears in the recurrence equation (10). The elements of the matrix $[M]$ is defined in terms of the constants E, F, and G as follows:

$$[M] = \begin{bmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{bmatrix}$$

where

$$M_1 = \frac{F_2G_3 - G_2F_3}{N} \qquad M_5 = \frac{E_1G_3 - G_1E_3}{N}$$

$$M_2 = \frac{G_2E_3 - E_2G_3}{N} \qquad M_6 = \frac{F_1E_3 - E_1F_3}{N}$$

$$M_3 = \frac{E_2F_3 - F_2E_3}{N} \qquad M_7 = \frac{F_1G_2 - G_1F_2}{N}$$

$$M_4 = \frac{G_1F_3 - F_1G_3}{N} \qquad M_8 = \frac{E_2G_1 - E_1G_2}{N}$$

$$M_9 = \frac{E_1F_2 - F_1E_2}{N}$$

where

$$N = E_1F_2G_3 + F_1G_2E_3 + G_1E_2F_3 - G_1F_2E_3 - E_1G_2F_3 - F_1E_2G_3$$

The matrices describing the variables and the integrals at times $m - 1$ to $m - 6$ form a set of rectangular matrices. These rectangular matrices consist of the elements defined by the constants E, F, and G. The matrices are given for each time interval as follows:

$$\left[S_1\right] = \begin{bmatrix} E_{18} & E_{24} & E_{30} & E_4 & E_5 & E_{12} & E_{11} & E_{36} \\ F_{18} & F_{24} & F_{30} & F_4 & F_5 & F_{12} & F_{11} & F_{36} \\ G_{11} & G_{14} & G_{17} & G_4 & G_5 & G_8 & G_{21} & G_{20} \end{bmatrix}$$

$$\left[S_2\right] = \begin{bmatrix} E_{19} & E_{25} & E_{31} & E_6 & E_{13} \\ F_{19} & F_{25} & F_{31} & F_6 & F_{13} \\ G_{12} & G_{15} & G_{18} & G_6 & G_9 \end{bmatrix}$$

$$\left[S_3\right] = \begin{bmatrix} E_{20} & E_{26} & E_{32} & E_7 & E_{14} \\ F_{20} & F_{26} & F_{32} & F_7 & F_{14} \\ G_{13} & G_{16} & G_{19} & G_7 & G_{10} \end{bmatrix}$$

$$\left[S_4\right] = \begin{bmatrix} E_{21} & E_{27} & E_{33} & E_8 & E_{15} \\ F_{21} & F_{27} & F_{33} & F_8 & F_{15} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[S_5\right] = \begin{bmatrix} E_{22} & E_{28} & E_{34} & E_9 & E_{16} \\ F_{22} & F_{28} & F_{34} & F_9 & F_{16} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[S_6\right] = \begin{bmatrix} E_{23} & E_{29} & E_{35} & E_{10} & E_{17} \\ F_{23} & F_{29} & F_{35} & F_{10} & F_{17} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

All the square and rectangular matrices which have been defined previously are for the stick-free condition. The recurrence formula given by equation (11), however, is a solution for the stick-fixed conditions and can be obtained by modifying the stick-free matrices. This modification consists of eliminating all the G constants and those E and F constants in the rectangular matrices which are post-multiplied by the variables in the column matrices that represent the acceleration, velocity, and displacement of the elevator. The matrices for the stick-fixed condition are written with a subscript zero and are defined as follows:

$$\left[M\right]_0 = \begin{bmatrix} \dfrac{F_2}{E_1F_2 - E_2F_1} & \dfrac{-E_2}{E_1F_2 - E_2F_1} \\ \\ \dfrac{-F_1}{E_1F_2 - E_2F_1} & \dfrac{E_1}{E_1F_2 - E_2F_1} \end{bmatrix}$$

$$\left[S_1\right]_0 = \begin{bmatrix} E_{18} & E_{24} & E_4 & E_5 & E_{11} \\ F_{18} & F_{24} & F_4 & F_5 & F_{11} \end{bmatrix}$$

$$\left[S_2\right]_0 = \begin{bmatrix} E_{19} & E_{25} & E_6 \\ F_{19} & F_{25} & F_6 \end{bmatrix}$$

$$\left[S_3\right]_0 = \begin{bmatrix} E_{20} & E_{26} & E_7 \\ F_{20} & F_{26} & F_7 \end{bmatrix}$$

$$\left[S_4\right]_0 = \begin{bmatrix} E_{21} & E_{27} & E_8 \\ F_{21} & F_{27} & F_8 \end{bmatrix}$$

$$\left[S_5\right]_0 = \begin{bmatrix} E_{22} & E_{28} & E_9 \\ F_{22} & F_{28} & F_9 \end{bmatrix}$$

$$\left[S_6\right]_0 = \begin{bmatrix} E_{23} & E_{29} & E_{10} \\ F_{23} & F_{29} & F_{10} \end{bmatrix}$$

APPENDIX C

FORCING FUNCTIONS

The forcing functions U_1, U_2, and U_3 are dependent upon the shape of the gust that the airplane is penetrating. If the unsteady-lift functions are known for the sharp-edge-gust condition, the evaluation of the forcing function is very simple and may be calculated for the sharp-edge-gust condition from the following equations:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}_m = \begin{Bmatrix} K_1 C_{L_{g_w}} \\ K_1 K_{25} C_{L_{g_w}} \\ 0 \end{Bmatrix}_m + \begin{Bmatrix} K_3 C_{L_{\epsilon_g}} \\ -K_3 K_{26} C_{L_{\epsilon_g}} \\ -K_{22}\epsilon_g \end{Bmatrix}_m + \begin{Bmatrix} K_2 C_{L_{g_t}} \\ -K_2 K_{26} C_{L_{g_t}} \\ -K_{15} \end{Bmatrix}_{m-\tau} \tag{C1}$$

where, in the recurrence formula, $\tau = \frac{l_{wt} + c}{c\,\Delta s}$ represents the time interval that the airplane has to travel before the tail enters the gust. For $m < \tau$, the terms having the subscript $m - \tau$ are neglected in the calculation of the forcing function. On the other hand, for $m \geqq \tau$ these terms are included. (The value of τ must be taken to the nearest integer.)

Because of the discontinuities of the functions ϵ_g and $C_{L_{\epsilon_g}}$ (see figs. 2 and 3), two values of the forcing function exist at both the points $m = l_{C_L} + \frac{0.75}{\Delta s}$ and $m = l_\epsilon + \frac{0.75}{\Delta s}$. In order to overcome this difficulty when evaluating the recurrence formula, two values of the forcing function are calculated at this point. The recurrence formula is evaluated at times $m = l_{C_L} + \frac{0.75}{\Delta s}$ and $m = l_\epsilon + \frac{0.75}{\Delta s}$ by use of the first value of the discontinuity and then reevaluated by use of the second value. Therefore, the recurrence formula gives two-valued solutions of the equations of motion at the points $m = l_{C_L} + \frac{0.75}{\Delta s}$ and $m = l_\epsilon + \frac{0.75}{\Delta s}$. In order to evaluate the formula at time $m + 1$ and at succeeding intervals, the first solution obtained from the recurrence formula is neglected, thereby the second one is used as the solution for succeeding calculations. The numerical solution to the equations of motion therefore contains discontinuities due to the functions ϵ_g and $C_{L_{\epsilon_g}}$. For most cases, however, the magnitude of l_ϵ and l_{C_L} are approximately equal; therefore, when these two distances are taken to the nearest integer of m, l_ϵ can be taken equal to l_{C_L} and only one discontinuity appears in the solution.

For an arbitrary gust shape, a number of Duhamel's integrals have to be evaluated for determining the forcing functions. These operations may be performed graphically or by various numerical methods shown in reference 5. The corresponding forcing functions for the arbitrary gust shape may be calculated from the following equations:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}_m = \begin{Bmatrix} K_1\Phi_w \\ K_1K_{25}\Phi_w \\ 0 \end{Bmatrix}_m + \begin{Bmatrix} K_3\Phi_{\epsilon_w} \\ -K_3K_{26}\Phi_{\epsilon_w} \\ -K_{22}\xi \end{Bmatrix}_m + \begin{Bmatrix} K_2\Phi_t \\ -K_2K_{26}\Phi_t \\ -K_{15}\dfrac{U}{U_{max}} \end{Bmatrix}_{m-\tau} \tag{C2}$$

where

$$\left.\begin{aligned} \Phi_w(s) &= \int_0^{s_1} C_{L_{g_w}}(s_1 - s)\frac{d\,\dfrac{U}{U_{max}}}{ds}\,ds \\ \Phi_t(s) &= \int_0^{s_1} C_{L_{g_t}}(s_1 - s)\frac{d\,\dfrac{U}{U_{max}}}{ds}\,ds \\ \Phi_{\epsilon_w}(s) &= \int_0^{s_1} C_{L_{\epsilon_g}}(s_1 - s)\frac{d\,\dfrac{U}{U_{max}}}{ds}\,ds \\ \xi(s) &= \int_0^{s_1} \epsilon_g(s_1 - s)\frac{d\,\dfrac{U}{U_{max}}}{ds}\,ds \end{aligned}\right\} \tag{C3}$$

If the response of the airplane for a number of arbitrary gust shapes is required, an alternate method would be far more expeditious. By first calculating the motions of the airplane for the sharp-edge-gust condition, the motions for each degree of freedom may be considered as indicial responses in Duhamel's integral. When the gust shape is utilized as the arbitrary forcing function, the response for each degree of freedom may be calculated by Duhamel's integral in a manner analogous to that used in calculating U_1, U_2, and U_3 for the arbitrary gust shape. Although the calculation of the response for the sharp-edge-gust condition would require additional work, the extra work is negligible when two or more responses to arbitrary gust shapes are desired.

REFERENCES

1. Jones, Robert T.: The Unsteady Lift of a Wing of Finite Aspect Ratio. NACA Rep. 681, 1940.

2. Donely, Philip, Pierce, Harold B., and Pepoon, Philip W.: Measurements and Analysis of the Motion of a Canard Airplane Model in Gusts. NACA TN 758, 1940.

3. Cohen, Doris: A Theoretical Investigation of the Rolling Oscillations of an Airplane with Ailerons Free. NACA Rep. 787, 1944.

4. Jones, Robert T., and Fehlner, Leo F.: Transient Effects of the Wing Wake on the Horizontal Tail. NACA TN 771, 1940.

5. Mazelsky, Bernard, and Diederich, Franklin W.: Two Matrix Methods for Calculating Forcing Functions from Known Responses. NACA TN 1965, 1949.

TABLE I

AIRPLANE PARAMETERS

$\left(\frac{dC_L}{d\alpha}\right)_w$, per radian	5.56
$\left(\frac{dC_L}{d\alpha}\right)_t$, per radian	3.21
S, sq ft	738
S_t, sq ft	275
ρ, slugs/cu ft	0.002378
V, ft/sec	308
η	0.85
$\frac{d\epsilon_1}{d\alpha}$	0.500
Aerodynamic center for wing and tail, percent M.A.C.	25
l_w, ft:	
For c.g. at $12\frac{1}{2}$ percent M.A.C.	-1.26
For c.g. at 25 percent M.A.C.	0
For c.g. at $37\frac{1}{2}$ percent M.A.C.	1.26
l_t, ft:	
For c.g. at $12\frac{1}{2}$ percent M.A.C.	36.64
For c.g. at 25 percent M.A.C.	35.38
For c.g. at $37\frac{1}{2}$ percent M.A.C.	34.12
l_{wt}, ft	25.98
U, ft/sec	50
g, ft/sec^2	32.2
I, slug-ft^2	209,600
c, ft	10
c_t, ft	7.58
W, lb	38,000

TABLE II

CALCULATED VALUES OF AIRPLANE STABILITY CONSTANTS INVOLVING ONLY THE STICK-FIXED CONDITION FOR CENTER OF GRAVITY AT 25 PERCENT M.A.C.

$K_1 = 1.9772$	$K_8 = 2.2272$
$K_2 = 0.36156$	$K_9 = -1.1136$
$K_3 = -0.18078$	$K_{10} = 0$
$K_4 = 0.041342$	$K_{11} = 7.8798$
$K_5 = 0.0075599$	$K_{12} = 0$
$K_6 = -0.0037799$	$K_{25} = 0$
$K_7 = 12.180$	$K_{26} = 0.0067616$

TABLE III

TRANSFORMED UNSTEADY-LIFT FUNCTIONS DUE TO UNIT JUMP OF ANGLE OF ATTACK ON WING AND TAIL FOR INFINITE ASPECT RATIO FOR INTERVAL $\Delta s = 1.5$ CHORDS

$B(s) = 1 - C_{L_{\alpha_w}}(s)$	$C(s) = 1 - C_{L_{\alpha_t}}(s)$
$B_0 = 0.500$	$C_0 = 0.500$
$B_1 = 0.280$	$C_1 = 0.240$
$B_2 = 0.181$	$C_2 = 0.147$
$B_3 = 0.133$	$C_3 = 0.106$
$B_4 = 0.105$	$C_4 = 0.084$
$B_5 = 0.088$	$C_5 = 0.069$
$B_6 = 0.075$	$C_6 = 0.057$

TABLE IV

VALUES OF E AND F ELEMENTS IN MATRICES $[M]_0$ AND $[S]_0$ AS A FUNCTION OF VARIABLE m

FOR AN AIRPLANE WITH CENTER OF GRAVITY AT 25 PERCENT M.A.C.

E and F	m = 1	m = 2	m = 2[a]	m = 3	m = 4	m = 5[b]
E_1	1.0156	1.0156	1.0131	1.0131	1.0131	1.0131
F_1	-.000018246	-.000018246	-.0000015975	-.0000015975	-.0000015975	-.0000015975
E_2	-5.0506	-5.0506	-4.6426	-4.6426	-4.6426	-4.6426
F_2	1.019674	1.019674	1.016916	1.016916	1.016916	1.016916
E_4	-.049499	-.049499	-.049499	-.045122	-.045122	-.045122
F_4	.000055155	.000055155	.000055155	.000025559	.000025559	.000025559
E_5	21.908	21.908	21.908	19.651	19.651	19.651
F_5	-.070717	-.070717	-.070717	-.055458	-.055458	-.055458
E_6	-7.0883	-7.0883	-7.0883	-7.0883	-6.6853	-6.6853
F_6	.0072856	.0072856	.0072856	.0072856	.0045608	.0045608
E_7	-2.8010	-2.8010	-2.8010	-2.8010	-2.8010	-2.8816
F_7	.0021800	.0021800	.0021800	.0021800	.0021800	.002725
E_8	-1.4655	-1.4655	-1.4655	-1.4655	-1.4655	-1.4655
F_8	.0012620	.0012620	.0012620	.0012620	.0012620	.0012620
E_9	-2.4493	-2.4493	-2.4493	-2.4493	-2.4493	-2.4493
F_9	.0020662	.0020662	.0020662	.0020662	.0020662	.0020662
E_{10}	-.52025	-.52025	-.52025	-.52025	-.52025	-.52025
F_{10}	.000858390	.000429195	.000429195	.000429195	.000429195	.000429195
E_{11}	14.583	14.583	14.583	13.294	13.294	13.294
F_{11}	-.016249	-.016249	-.016249	-.0075297	-.0075297	-.0075297
E_{18}	-.0319969	-.0319969	-.0319969	-.026799	-.026799	-.026799
F_{18}	.000040940	.000040940	.000040940	.0000057945	.0000057945	.0000057945
E_{19}	.024060	.024060	.024060	.024060	.022692	.022692
F_{19}	-.000024730	-.000024730	-.000024730	-.000024730	-.000015481	-.000015481
E_{20}	.0095075	.0095075	.0095075	.0095075	.0095075	.0097811
F_{20}	-.0000073998	-.0000073998	-.0000073998	-.0000073998	-.0000073998	-.0000092496
E_{21}	.0049744	.0049744	.0049744	.0049744	.0049744	.0049744
F_{21}	-.0000042836	-.0000042836	-.0000042836	-.0000042836	-.0000042836	-.0000042836
E_{22}	.0083134	.0083134	.0083134	.0083134	.0083134	.0083134
F_{22}	-.0000070133	-.0000070133	-.0000070133	-.0000070133	-.0000070133	-.0000070133
E_{23}	.0017658	.0017658	.0017658	.0017658	.0017658	.0017658
F_{23}	-.00000145685	-.00000145685	-.00000145685	-.00000145685	-.00000145685	-.00000145685
E_{24}	11.036	11.036	11.036	10.174	10.174	10.174
F_{24}	-.044057	-.044057	-.044057	-.038233	-.038233	-.038233
E_{25}	-5.0555	-5.0555	-5.0555	-5.0555	-4.8288	-4.8288
F_{25}	.026141	.026141	.026141	.026141	.024608	.024608
E_{26}	-.89203	-.89203	-.89203	-.89203	-.89203	-.93737
F_{26}	.0076401	.0076401	.0076401	.0076401	.0076401	.0073335
E_{27}	-.66033	-.66033	-.66033	-.66033	-.66033	-.66033
F_{27}	.0044649	.0044649	.0044649	.0044649	.0044649	.0044649
E_{28}	-.10811	-.10811	-.10811	-.10811	-.10811	-.10811
F_{28}	.0073100	.0073100	.0073100	.0073100	.0073100	.0073100
E_{29}	-.224575	-.224575	-.224575	-.224575	-.224575	-.224575
F_{29}	.0015185	.0015185	.0015185	.0015185	.0015185	.0015185

[a]Two values occur at m = 2 because of the effect of the function $C_{L_{\epsilon_\alpha}}$ on the tail.

[b]For all values of m greater than 5, use the values of E and F in the column for m = 5.

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TABLE V

CALCULATED FORCING FUNCTIONS U_1 AND U_2 FOR A SHARP-EDGE GUST

FOR AN AIRPLANE WITH CENTER OF GRAVITY AT 25 PERCENT M.A.C.

1	2	3	4	5	6	7	8	9
m	$K_1 C_{L_{g_{w_m}}}$	$K_2 C_{L_{g_{t_{m-\tau}}}}$	$K_3 C_{L_{\epsilon_{g_m}}}$	$(U_1)_m = 2 + 3 + 4$	K_{25} 2	$-K_{26}$ 3	$-K_{26}$ 4	$(U_2)_m = 6 + 7 + 8$
0	0	0	0	0	0	0	0	0
1	1.2456	0	.02856	1.2742	0	0	-.00019311	-.00019311
2	1.5323	0	.02856	1.5609	0	0	-.00019311	-.00019311
2	1.5323	0	-.18078	1.3515	0	0	.0012224	.0012224
3	1.6628	.24980	-.18078	1.7318	0	-.0016890	.0012224	-.0004666
4	1.7301	.29724	-.18078	1.8466	0	-.0020098	.0012224	-.0007874
5	1.7775	.31626	-.18078	1.9130	0	-.0021384	.0012224	-.0009160
6	1.8111	.32692	-.18078	1.9572	0	-.0022105	.0012224	-.0009881
7	1.8388	.33433	-.18078	1.9924	0	-.0022606	.0012224	-.0010382
8	1.8605	.34001	-.18078	2.0197	0	-.0022990	.0012224	-.0010766
9	1.8803	.34442	-.18078	2.0439	0	-.0023288	.0012224	-.0011064
10	1.8922	.34797	-.18078	2.0594	0	-.0023528	.0012224	-.0011304
11	1.9040	.35075	-.18078	2.0740	0	-.0023716	.0012224	-.0011492
12	1.9129	.35295	-.18078	2.0851	0	-.0023865	.0012224	-.0011641
13	1.9219	.35473	-.18078	2.0959	0	-.0023985	.0012224	-.0011761
14	1.9278	.35614	-.18078	2.1032	0	-.0024081	.0012224	-.0011857
15	1.9327	.35726	-.18078	2.1092	0	-.0024156	.0012224	-.0011932

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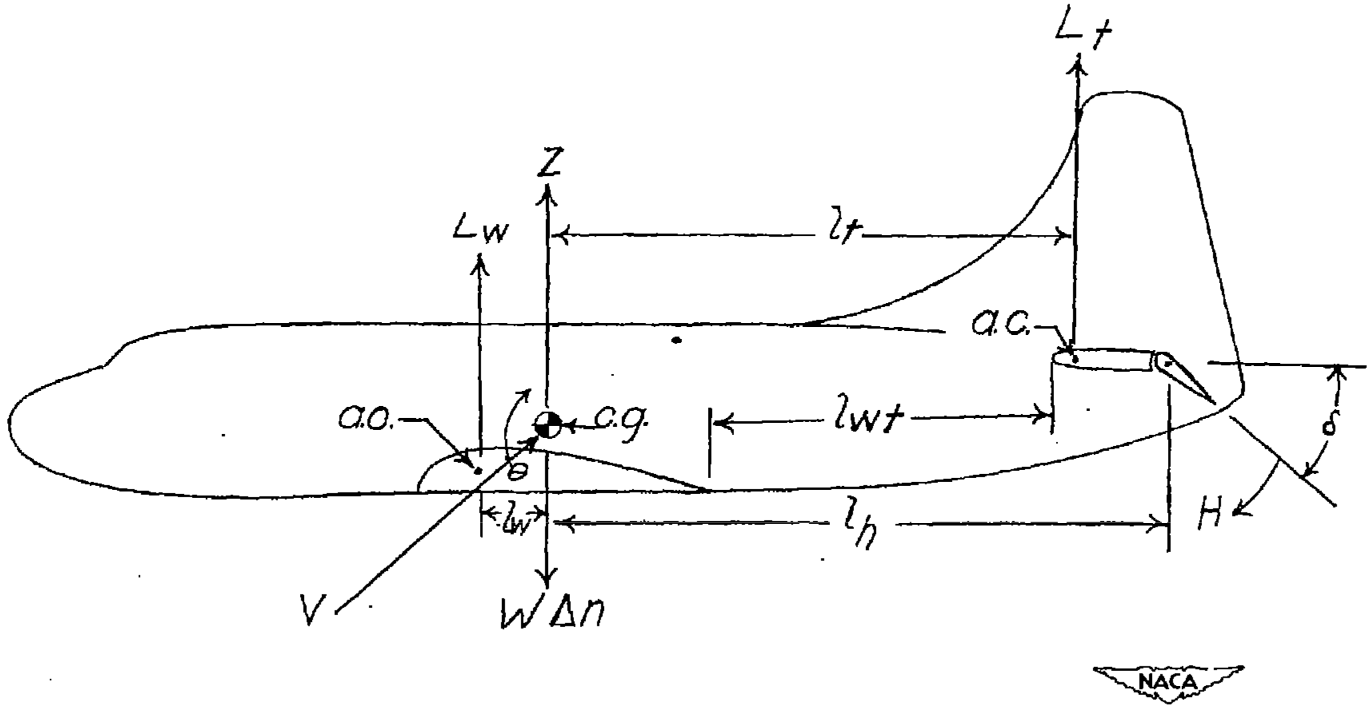
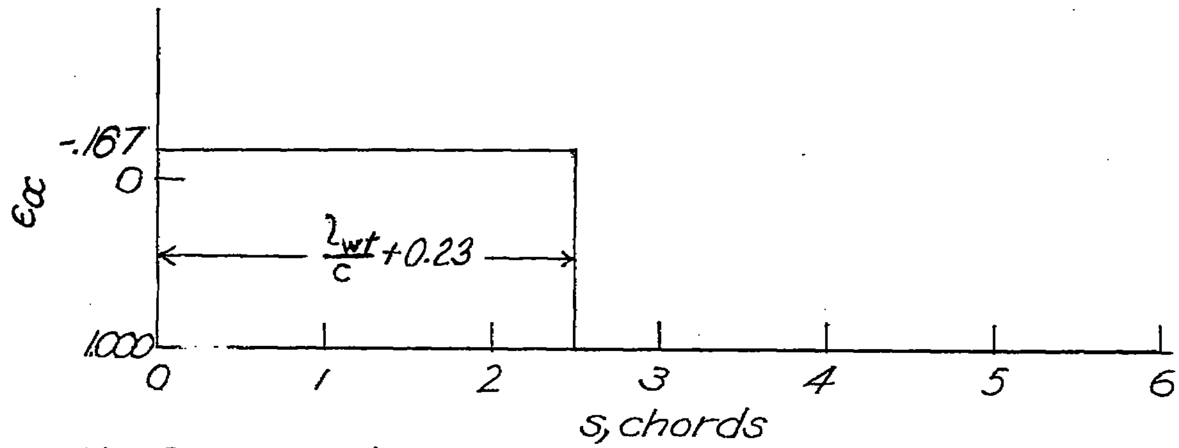
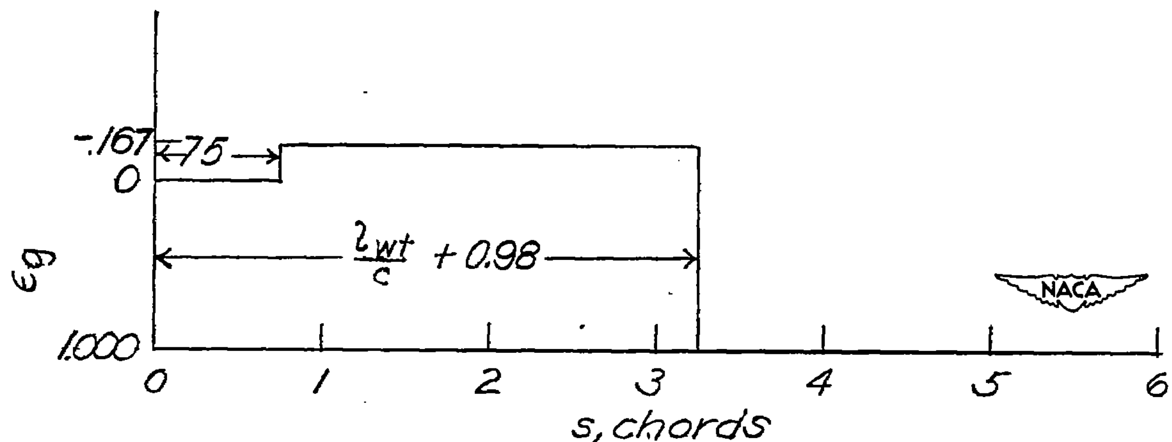


Figure 1.- Forces and moments assumed acting on airplane during penetration of a gust. (Distances, forces, and moments shown in postive direction)



(a) Due to unit jump of angle of attack on wing.



(b) Due to penetration of wing into a sharp-edge gust.

Figure 2 .- Normalized unsteady-downwash angle at horizontal tail.

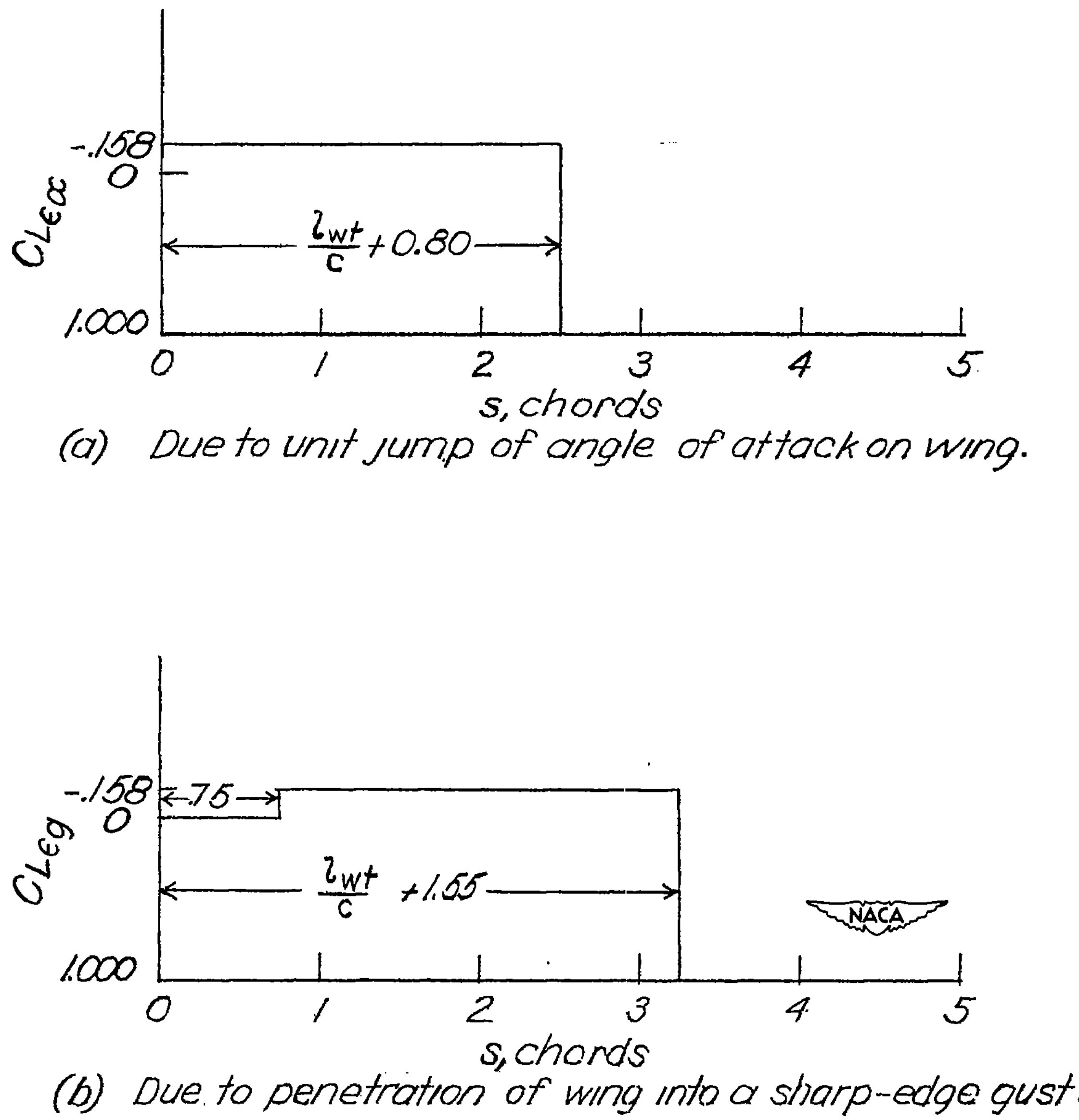


Figure 3. - Effective normalized unsteady-lift function on the tail.

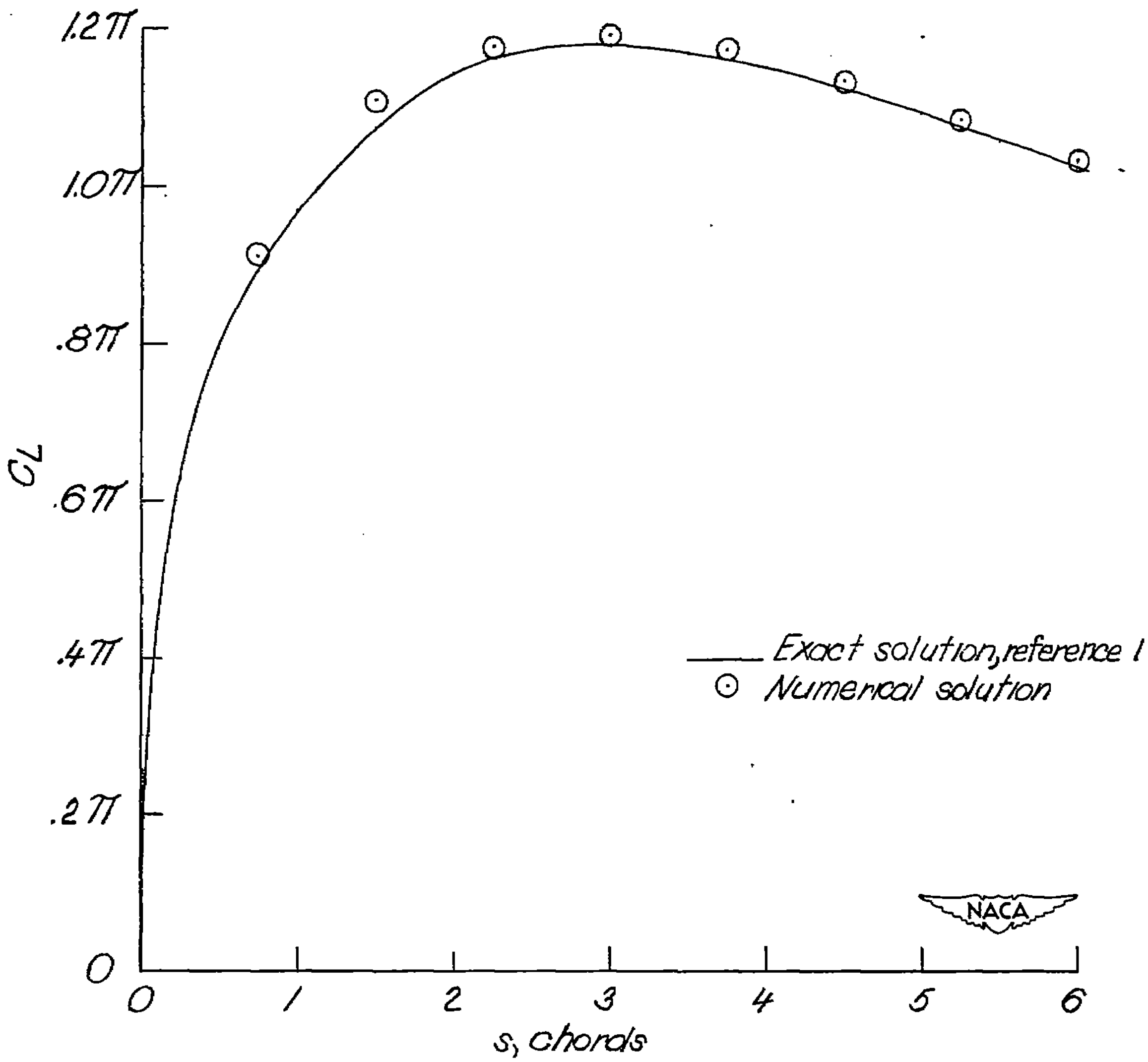


Figure 4.- Comparison of numerical and exact solutions of the lift coefficient on a wing penetrating a sharp-edge gust. $\mu_g = 13.2$.

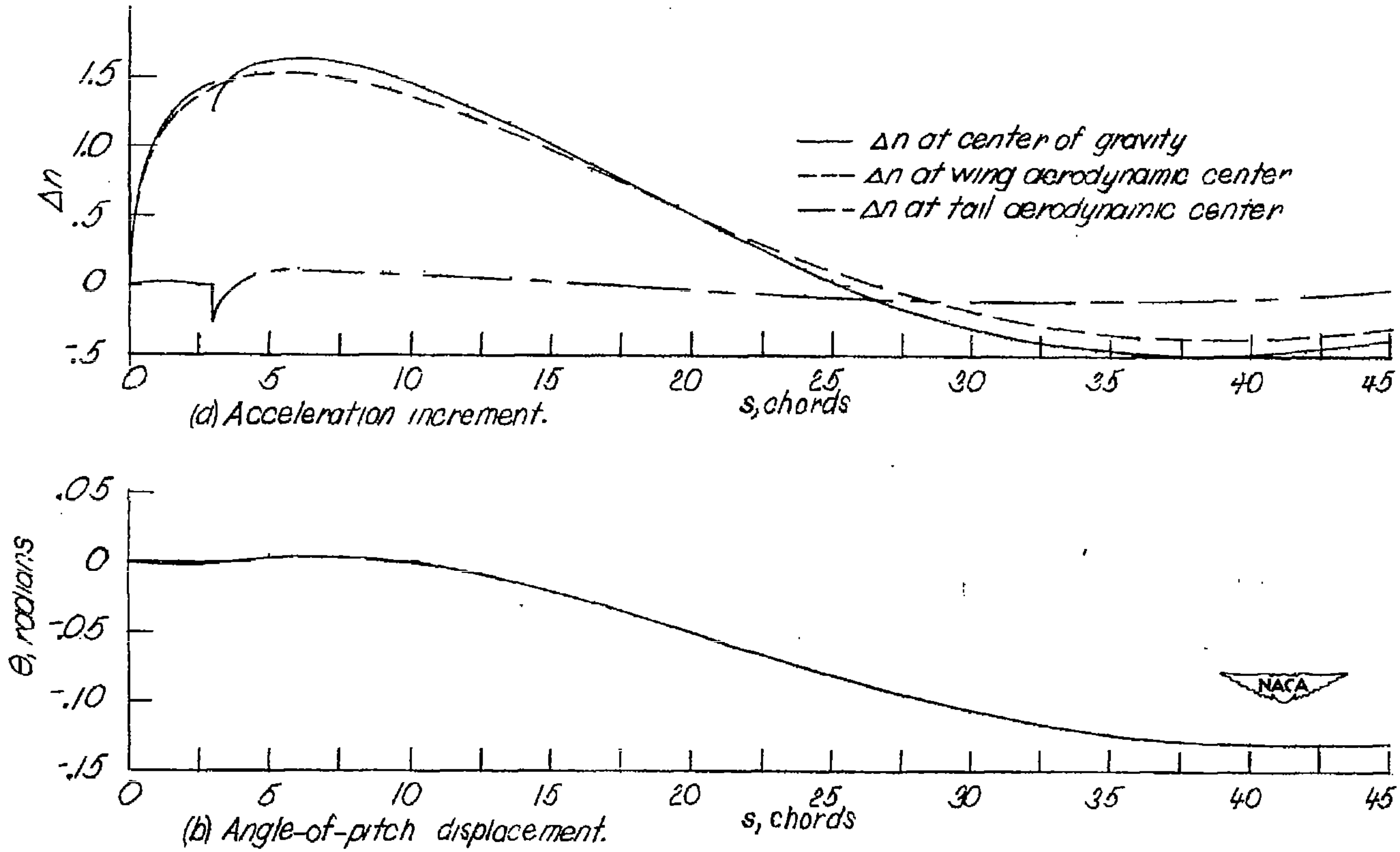


Figure 5.- Typical responses for the penetration of a sharp-edge gust with an average center-of-gravity position.

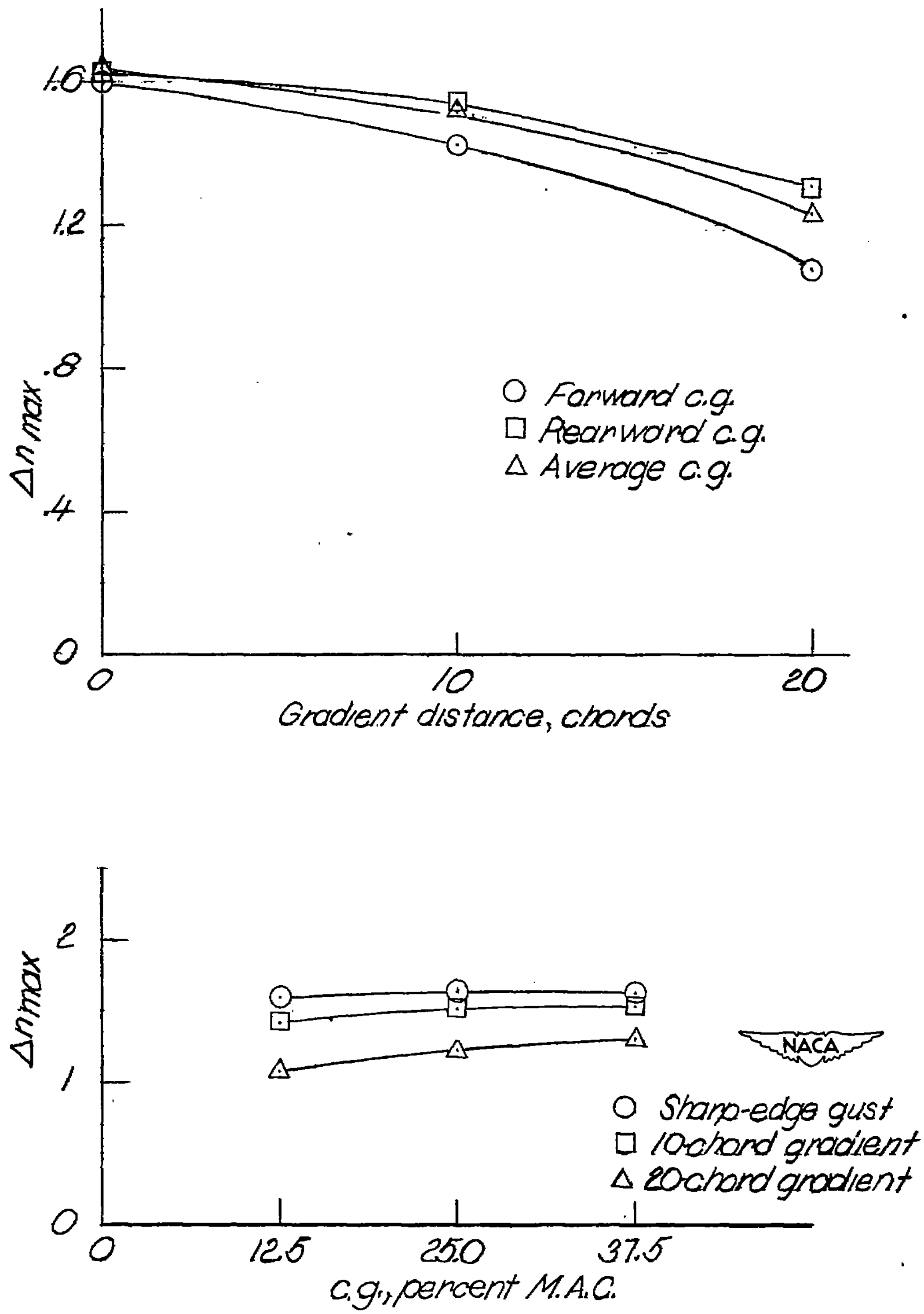


Figure 6.-Effect of gust shape and center-of-gravity position on gust load factor.

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